Testing Orthogonal Shapes*

Kathleen Romanik†
Department of Computer Science
University of Maryland
College Park, MD 20742
romanik@cs.umd.edu

Steven Salzberg‡
Department of Computer Science
Johns Hopkins University
Baltimore, MD 21218
salzberg@cs.jhu.edu

1 Introduction

Many tasks in the physical world require that objects be distinguished strictly through the sense of touch. For example, a robot on a manufacturing assembly line must pick a specific part out of a bin using only touch to identify the correct part. A robot designed to perform quality assurance on a manufactured part probes each dimension using touch to check that it falls within specified tolerances. In both cases, efficiency is of the utmost importance, so the robot should make a minimal number of probes.

The idea of using probes to identify or test geometric objects has been explored by various people in the area of geometric probing [Ber86, CY87, RS90, Ski88]. Although there are many different kinds of geometric probes, the type of probe we use in this paper is a "point probe" [RS90]. The input to this probe is a point in Euclidean space and the output is either "positive", if the point is inside the object being probed, or "negative" if it is outside.

The idea of using point probes to distinguish objects is closely related to the "helpful teacher" learning model [SDHK91, GK91] developed in the area of machine learning. Machine learning algorithms attempt to infer a concept description from a set of examples, and in the helpful teacher model these examples are produced by a teacher who knows the concept and is attempting to teach it. The examples produced by the teacher are analogous to the test points (or point probes) produced by a testing algorithm.

In this paper we describe testing algorithms for two different classes of objects: sets of disjoint rectangles in 2-D and higher dimensions, and general orthogonal shapes in 2-D and 3-D.

*A complete version of this paper appears as a Johns Hopkins University Department of Computer Science technical report, JHU-91/23, December 1991.

†Supported in part by NSF Grant CCR-9112976 and ONR Grant N00014-92-J-1254.

‡Supported in part by NSF Grant IRI-9116843.

2 Testability

An object is a subset of Euclidean space, E^d; in particular, it is a subset of a unit box in d dimensions, since any object can be scaled down until it is contained in a unit box. An *object class* is a set Q of objects. In this paper we examine only orthogonal shapes; i.e., polyhedral in which all edges are parallel to the axes.

Given an object $p \in Q$ to be tested and a target object (or model) $q \in Q$, p is *consistent* with q on some finite set of test points $t = \{t_1, t_2, \ldots, t_m\}$ if p and q contain the same subset of t, i.e.; $t_i \in q$ iff $t_i \in p$ for $1 \leq i \leq m$. The *error* of p with respect to q is given by $V(q \Delta p)$, where $V(p)$ denotes the d-dimensional volume of an object p and $q \Delta p$ denotes the symmetric difference of the sets. Thus, the error of the object p is measured as the volume of the region that forms the symmetric difference between p and q.

Definition. T is a *testing algorithm for Q with test set size m* if for all $\epsilon \in (0,1)$ and for all $q \in Q$, T produces a finite set of points $T(q,\epsilon)$ in E^d of size no greater than $m(\epsilon)$ and these points have the property that for all $p \in Q$, if p is consistent with q on $T(q,\epsilon)$, then $V(q \Delta p) \leq \epsilon$. $T(q,\epsilon)$ is called a *test set for q with respect to the class Q*. For each $t_i \in T(q,\epsilon)$, if $t_i \in q$ then t_i is a *positive test point*; otherwise, t_i is a *negative test point.*

Thus given a target object $q \in Q$ and an error bound $\epsilon \in (0,1)$, T produces a test set for q such that any consistent object p has error no more than ϵ. If such a T and m exist, then the class Q is *testable with test set size m*. If T produces a constant size test set (i.e. if m is a constant k), then Q is *k-testable.*

3 Disjoint Rectangles

As a starting point, we consider orthogonal shapes that consist of a fixed number of disjoint hyperrectangles. In [RS90, GK91] it was shown that one d-dimensional orthogonal hyperrectangle can be tested optimally with $2d+2$ test points. This result can be generalized in E^2.

Theorem 1. Let R_n^2 be the class consisting of objects composed of n disjoint rectangles in E^2. R_n^2 is $6n$-testable.

Proof. Given $q \in R_n^2$ and $0 < \epsilon < 1$, let $w_{\min}$ be the width of the narrowest rectangle (in either the x or the y dimension) and let $w_{\max}$ be the width of the widest rectangle in q. Also, let each rectangle $r \in q$ be represented by its minimum and maximum corners, i.e. $r = ((x_{\min}, y_{\min}), (x_{\max}, y_{\max}))$. Let $\text{perp}_{\min}$ be the smallest perpendicular distance between 2 rectangles in q, and let $\alpha = \frac{1}{2}\min(w_{\min}, \text{perp}_{\min}, \frac{\epsilon}{4nw_{\max}})$. For each rectangle $r \in q$ choose 2 positive test points, one which is a distance α in each direction from the minimum corner of r (called a lower point) and one which is α from the maximum corner of r (called an upper point). Next choose 4 negative test points by reflecting each positive test point outside the rectangle in each direction by a distance of α. See Figure 1 for an illustration.

Claim 1a. No rectangle r that is consistent with q on the selected points contains 3 or more positive points. (See [RS91] for a proof.)

Since no rectangle consistent with q on the selected points contains 3 or more positive points, the only way to divide the $2n$ positive points among n consistent rectangles is for each rectangle to contain exactly 2 positive points. Now we must show that the only way to do this is

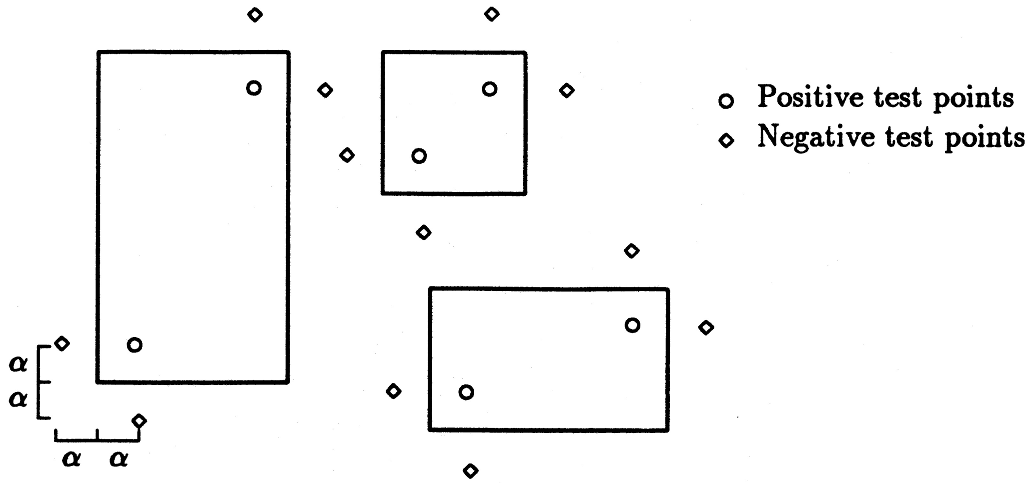


Figure 1: Set of disjoint rectangles with test points

to partition the points the same way q does, for which we need the notion of a "top" rectangle. For a set of n disjoint rectangles, we define a rectangle $r = ((x_{\min}, y_{\min}), (x_{\max}, y_{\max}))$ as a *top rectangle* if the quadrant that has $(x_{\min}, y_{\min})$ as its minimum point contains no part of any other rectangle in the set. We show next that a top rectangle always exists.

Claim 1b. Any set of n disjoint rectangles contains a top rectangle.

Proof. Define a directed graph G which has a vertex for each rectangle in the set and an edge $r_i \rightarrow r_j$ for each pair of rectangles such that r_j is partially contained in the quadrant defined by the minimum point of r_i. The edge relation on G is antisymmetric since if $r_i \rightarrow r_j$ and $r_j \rightarrow r_i$ were both edges in G, then the rectangles r_i and r_j would not be disjoint. Also, G is acyclic. If it were not, then it would contain a minimal length cycle $r_1 \rightarrow r_2 \rightarrow r_3 \ldots \rightarrow r_k \rightarrow r_1$ of length at least three. Since the rectangles are all disjoint, r_1 and r_2 must be nonoverlapping in at least one dimension; w.l.g. say the x dimension. This means $x_{1,\max} < x_{2,\min}$, since $r_1 \rightarrow r_2$ is an edge. Similarly, r_2 and r_3 must be nonoverlapping in the x dimension (i.e. $x_{2,\max} < x_{3,\min}$), or a contradiction results. That is, an overlapping x dimension would imply $y_{2,\max} < y_{3,\min}$ (since r_2 and r_3 would not overlap in the y dimension), but then (since edge $r_1 \rightarrow r_2$ implies $y_{1,\min} < y_{2,\max}$) $y_{1,\min} < y_{2,\max} < y_{3,\min} < y_{3,\max}$ and $x_{1,\min} < x_{1,\max} < x_{2,\min} < x_{3,\max}$, so $r_1 \rightarrow r_3$ would be an edge in G which contradicts the assumption that the cycle chosen was of minimal length. Using the same argument, all rectangles in this cycle must be nonoverlapping in the x dimension. Therefore, we have $x_{1,\min} < x_{1,\max} < x_{2,\min} \cdots < x_{k,\max} < x_{1,\min}$, which is a contradiction, so G must be acyclic. Since G is acyclic and its set of vertices is finite, there exists a rectangle with outdegree 0 in G. This rectangle is a top rectangle. Claim 1b □

Claim 1c. Any set of n disjoint rectangles consistent with q on the selected points partitions the positive points the same way as q does.

Proof Sketch. Any consistent rectangle containing two positive points including the minimum positive point of some top rectangle $r \in q$ must also contain r's maximum positive point. By induction, the remaining $n-1$ rectangles must partition the remaining positive points in the same way as q. Claim 1c □

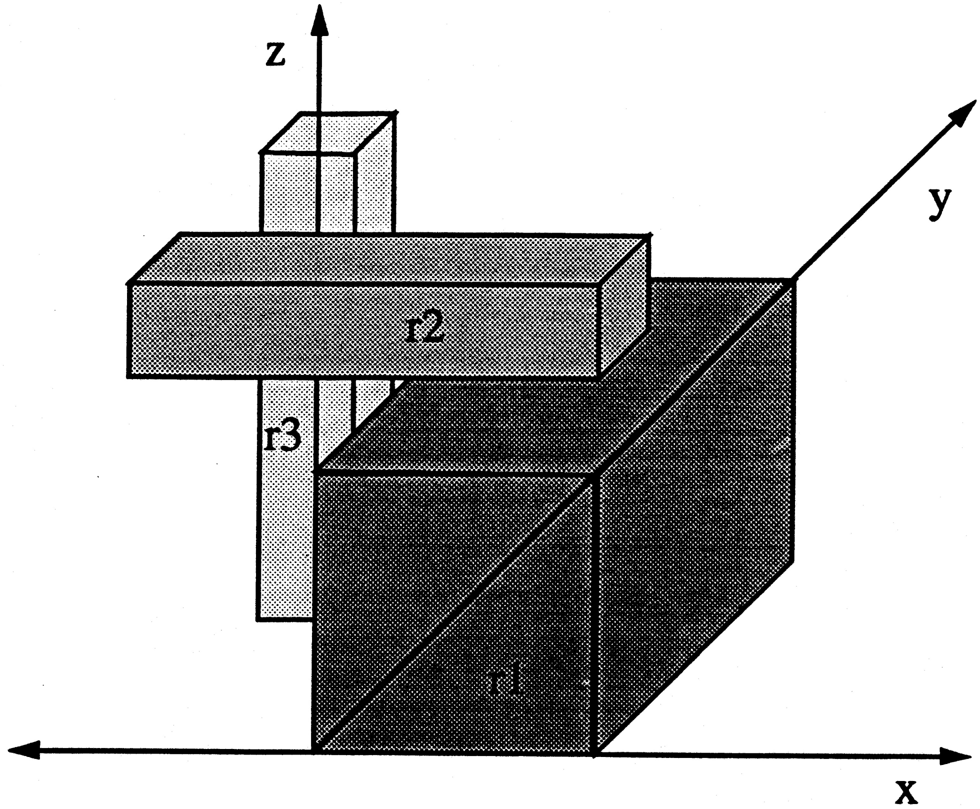


Figure 2: Set of disjoint rectangles without a top rectangle

Now we return to the proof of Theorem 1. Since a set of n disjoint rectangles that is consistent with q on the selected points must partition the positive points the same way as q does, the error of this set is bounded by the test points around each rectangle. One consistent rectangle differs from the corresponding rectangle of q by no more than $4\alpha w_{\max} + 4\alpha^2$, so the total error is no more than $4n\alpha(w_{\max} + \alpha)$, which by the choice of α is less than ϵ. □

4 Higher Dimensional Objects

Surprisingly, the proof used for Theorem 1 cannot be generalized to d dimensions. This is because the notion of a top rectangle does not generalize to higher dimensions. In fact, even the method of using test points at opposite corners does not work in general to test n disjoint rectangles in d dimensions.

Figure 2 illustrates a configuration of three rectangles, $r_1 = ((0,0,0),(3,4,3)), r_2 = ((-2,0,4),(3,1,5)), r_3 = ((-2,2,0),(-1,3,5))$, for which there is no top rectangle. This configuration cannot be tested using points near the minimum and maximum corner of each rectangle, since the three rectangles $r_1' = ((0,0,0),(3,1,5)), r_2' = ((-2,0,4),(-1,3,5)), r_3' = ((-2,2,0),(3,4,3))$ will be consistent with it on these points but will have error greater than ϵ for small enough ϵ.

Guibas and Yao [GY80] showed a similar result by proving that given any set of disjoint orthogonal rectangles in the plane and a direction vector θ, there exists an ordering of the rectangles such that for any distance d, moving the rectangles one by one according to the ordering by d in the direction of θ will not cause any rectangle to intersect another one. However, they show that such an ordering does not always exist for three dimensional orthogonal rectangles.

Fortunately, we can still test d-dimensional disjoint rectangles with $O(nd)$ test points using a different algorithm.

Theorem 2. Let R_n^d be the class consisting of objects composed of n disjoint rectangles in E^d. R_n^d is $n(4d+2)$-testable.

Test points for a target object in R_n^d are chosen in a similar manner to the proof of Theorem 1. Extra negative test points are chosen by reflecting each positive test point outside the object in $2d$ directions. The extra negative test points insure that no consistent rectangle contains positive points from more than one rectangle of the target. The proof is quite similar to that of Theorem 1. For details see [RS91].

5 General Orthogonal Shapes

In this section we examine how to test general orthogonal shapes that might contain holes. Some restrictions must be made on objects belonging to the class being tested. For orthogonal shapes we use the restriction that all objects in the set have the same number of corners, where a *corner* for a d-dimensional object is defined as the intersection point of d or more $(d-1)$-dimensional boundary faces of the object. Also we assume that no objects contain degenerate boundary faces. A boundary face f of an orthogonal object is an *exterior degenerate face (interior degenerate face)* if there exists a one dimensional line l that has a perpendicular intersection with f and there exists a distance $\delta > 0$ such that for all ϵ, $0 < \epsilon < \delta$, the two points on l that are a distance ϵ from f are both outside (inside) the object.

First we consider orthogonal objects in two dimensions. There are three ways a corner can result from the intersection of two lines in E^2. If only one of the four quadrants formed by the intersection is interior to the object, then it is a *convex corner*. If three quadrants are interior, then it is a *concave corner*. If two diagonal quadrants are interior, then we consider it to be the meeting point of two convex corners (see Figure 3).

Theorem 3. Let O_n^2 be the class consisting of orthogonal objects (with or without holes) in E^2 with n corners. O_n^2 is $3n$-testable.

Proof sketch. Given $o \in O_n^2$ and $0 < \epsilon < 1$, let $l_{\min}$ be the length of the shortest side of o, let $l_{\max}$ be the length of the longest side of o, let $d_{\min}$ be the minimum distance between any two parallel, non-colinear sides of o, and let $d_{\max}$ be the maximum distance between any two parallel sides of o. Let $\alpha = \frac{1}{2}\min(l_{\min}, d_{\min}, \frac{2\epsilon}{n(l_{\max}+d_{\max}-l_{\min})})$. For each convex corner of o, choose one positive test point a distance of α in each direction from the corner, and choose two negative test points by reflecting the positive test point outside the object in each direction by a distance of α. Similarly, for each concave corner choose one negative test point and two positive test points. See Figure 3 for an illustration of an object in O_{16} with its test points.

This method of choosing test points will yield no more than $3n$ points. To show that any other object in O_n^2 that is consistent on the given test points is within the error bound of ϵ, we first show that any consistent object must contain a corner "close" to every corner of o. Next we argue that by constraining the placement of its corners, a consistent object can only differ from the target by a slight expansion and by the presence of narrow strips between aligned corners of the target. The width of these strips and the amount of expansion are determined by the choice of α, so the error of a consistent object can be made less than ϵ. For the remaining details of the proof, see [RS91]. □

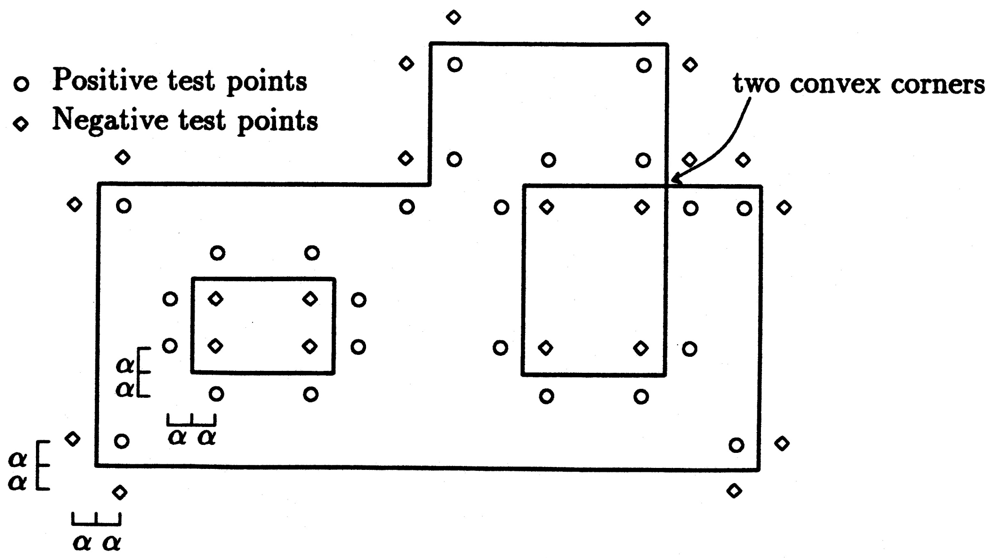


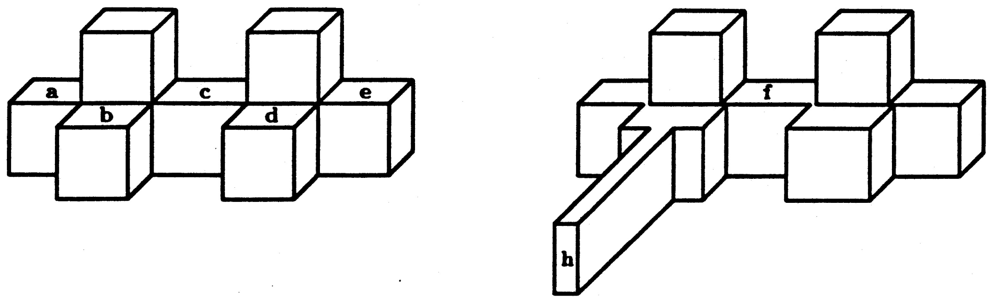
Figure 3: A 2-D object in O_{16} with its test points

When we consider testing three dimensional orthogonal objects, the task becomes more difficult. First, there are more than two types of corners in 3D. Also, further restrictions must be made on the definition of a face. In addition to not being degenerate, we make the restriction that a face cannot pass through an edge or a corner of an object. This eliminates the possibility that a face will have two orientations (i.e., that the interior of the object will lie on one side of the face at one place and will lie on the other side of the face at another place). Since there is no direct correspondence between the number of faces of an object and the number of vertices, restricting a class of objects by the number of faces that an object may have is not sufficient for testing the class. That is, the class F_n of three dimensional orthogonal objects with n faces is not testable using a number of test points that is a polynomial in only n and not ϵ. For example, the shape in Figure 4(a) has 26 faces, where faces a, b, c, d and e are separate faces. No matter how close negative test points are chosen to the shape, a consistent object, such as the one in Figure 4(b), can be found that expands the sides of faces b and d so that the five faces a, b, c, d and e are merged into one face f. Since this consistent object now has four less faces than the target object, four additional faces can be added to it to form the protrusion ending with face h. For ϵ chosen sufficiently small, this protrusion will cause the consistent object to have error greater than ϵ.

Despite these difficulties, three dimensional orthogonal shapes can still be tested efficiently, if we use the number of corners to define the class of objects we want to test.

Theorem 4. Let O_n^3 be the class consisting of orthogonal objects in E^3 with n corners. O_n^3 is $8n$-testable.

For a target object in O_n^3, 8 test points are chosen around each corner – one test point in each quadrant that meets at the corner. These points insure that a consistent object contains a corner close to every true corner of the target. Therefore the consistent object can only differ from the target by a slight expansion and the presence of strips. See [RS91] for proof details.



(a) A target object in F_{26} (b) A consistent object in F_{26}

Figure 4: Test points cannot determine if faces a, b, c, d and e are merged

References

[Ber86] Herbert J. Bernstein. Determining the Shape of a Convex n-Sided Polygon By Using $2n+k$ Tactile Probes. *Information Processing Letters*, 22(5):255–260, April 28 1986.

[CY87] Richard Cole and Chee K. Yap. Shape from Probing. *Journal of Algorithms*, 8(1):19–38, March 1987.

[GK91] Sally A. Goldman and Michael J. Kearns. On the Complexity of Teaching. In Leslie G. Valiant and Manfred K. Warmuth, editors, *Proceedings of the Fourth Annual Workshop on Computational Learning Theory*, pages 303–314, Santa Cruz, CA, August 5-7 1991. Morgan Kaufmann Publishers.

[GY80] Leo J. Guibas and F. Frances Yao. On Translating a Set of Rectangles. In *Proceedings of the Twelfth Annual ACM Symposium on Theory of Computing*, pages 154–160, Los Angeles, CA, April 28-30 1980.

[RS90] Kathleen Romanik and Carl Smith. Testing Geometric Objects. In Jorge Urrutia, editor, *Proceedings of the Second Canadian Conference in Computational Geometry*, pages 14–19, Ottawa, Ontario, Canada, August 6-10 1990. University of Ottawa.

[RS91] Kathleen Romanik and Steven Salzberg. Testing Orthogonal Shapes. Technical Report JHU-91/23, Johns Hopkins University, December 1991.

[SDHK91] Steven Salzberg, Arthur Delcher, David Heath, and Simon Kasif. Learning with a Helpful Teacher. In *Proceedings of the Twelfth International Joint Conference on Artificial Intelligence*, pages 705–711, Sydney, Australia, August 1991. Morgan Kaufmann Publishers.

[Ski88] Steven S. Skiena. *Geometric Probing*. PhD thesis, University of Illinois at Urbana-Champaign, Urbana, IL, April 1988.