On the recovery of a polygon's shape from its diameter function*

(extended abstract)

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Abstract

In robot applications, a parallel-jaw gripper or scanning light beam can be used to measure the *diameter* of a part as a function of the angle of grasp or projection. Can such measurements be used to determine part shape? Our primary result is that the part shape cannot be uniquely recovered from its diameter function even when we are restricted to the class of planar polygonal parts. We show that for any polygonal part, there exists an infinite class of parts with the same diameter function. Since most of these parts have parallel edges, we consider the problem of identifying a representative part having no parallel edges from this class. We show that deciding if such a part exists is *NP*-Complete.

1 Introduction

In this paper we consider the class of parts with constant polygonal cross section or planar polygonal parts. Our objective is to recover the shape of a part's cross section by grasping the part with a parallel-jaw gripper at various angles and measuring the distance between the jaws (the diameter at that angle of measurement). We can also measure the diameter as the length of the projection from a scanning light beam. Both of these sensors are inexpensive and widely available. Can such measurements be used to determine part shape? This problem may thus be posed in the context of geometric probing [1, 16] as a question of shape recovery from "diameter probes", which are weaker than projection probes [9] in that they return only the *length* of the projection.

Let S^1 denote the space of planar orientations $[0, 2\pi)$. Given a fixed polygon P in an $x-y$ coordinate frame, the diameter function $d : S^1 \rightarrow \mathcal{R}_+$ of P is formally defined as follows. Imagine two (infinite) parallel lines l, h (supporting lines) both making angle ϕ with the x-axis, just touching P so that P lies entirely in the region between the two lines. In such a case we say that the supporting lines l, h are at *orientation* ϕ with respect to the (fixed) polygon P. The distance between the two lines that are at orientation ϕ is $d(\phi)$.

The diameter function has period π. Also, diameter function of a polygon is the diameter function of its convex hull and so we can only seek shape recovery of the convex hull of a polygon. Therefore we may assume all polygons considered in this paper to be convex.

Our results in this paper are the following. Call a diameter function valid if it is the diameter function of a polygon. In section 2 we show that for every valid diameter function d there exist infinitely many polygons consistent with it. Thus, complete shape recovery of a polygon from its diameter function is impossible. However, we show that the orientation of every edge of the polygon and partial perimeters of the polygon along any orientation are recoverable from the diameter function. Looking at the proofs of these results, it becomes natural to consider shape recovery for the restricted class of polygons having no parallel edges (we call such polygons *Minimal polygons*). In section 3 we show that although the number of minimal polygons consistent with a given diameter function is finite, complete shape recovery still impossible. Also we show that deciding the question, "Given valid diameter function d, is there a minimal polygon consistent with d" to be *NP*-complete.

The proofs of many of the propositions in this extended abstract are omitted due to space limitations and can be found in [13].

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Related Work: The concept of *diameter* of a set of points, the maximizing distance over all pairs of points, is well studied in computational geometry [11]. Diameter functions have been alternatively termed "width functions" in [17] and were applied by Jameson [6] to determine grasp stability for a part grasped in the jaws of a parallel-jaw gripper. Goldberg [4] used the diameter function to generate plans, in $O(n^2)$ time, to orient n-gonal parts. Rao and Goldberg [14] extend these results to curved parts. Prasanna and Rao [10] study parallel mesh algorithms for orienting parts. [8] investigates tactile exploration of objects using a parallel jaw gripper. A major lemma (Lemma 3) in proving our *NP*-completeness result is showing that the problem of arranging (translating) a set of line segments, no two of which are parallel, into a convex polygon is *NP*-complete. This bears some resemblance to the result of Rappaport [15] which shows that the problem of drawing (additional) line segments to connect a collection of given fixed line segments (by their endpoints) into a simple circuit is *NP*-complete. In our problem, we allow the segments to translate (only) and we do not allow additional line connecting segments. Dekster [2] considers the problem of assembling any r out of n segments into an r-gon and gives a rule for this to be true on basis of the lengths of the segments. However, in "assembling" he allows rotations (in addition to translations).

1.1 Valid diameter functions

In this section we characterize valid diameter functions. A function $f : S^1 \rightarrow \mathcal{R}_+$ is said to be a "good" piecewise sinusoidal function (gpsf) if there exists a finite integer $Z \geq 4$, and a cyclic ordering of orientations

$$\phi_0 < \phi_1 < \ldots < \phi_{Z-1} < \phi_Z = \phi_0$$

such that $\forall j \in \{0, 1, \ldots, Z-1\}$, and $\forall \phi \in [\phi_j, \phi_{j+1}]$: $f(\phi) = l_j \cos(\phi + \alpha_j)$, for some $l_j \in \mathcal{R}_+$, and $\alpha_j \in S^1$.

Notice that a gpsf is continuous, single valued, and has a finite number of local extrema. Furthermore, a gpsf is differentiable at all but a finite number (at most Z) of orientations in S^1. For a gpsf f, let $MAX(f), MIN(f)$ denote circularly ordered list of local maxima, local minima orientations, respectively. $\Phi(f)$, the *set of transition orientations* of f, denotes the circularly ordered list $\{\phi_0, \ldots, \phi_{Z-1}\}$ from the definition of the gpsf f.

Theorem 1 *A function f is a valid diameter function if and only if*

1. f is a gpsf,

2. f has period π, and

3. $\text{MIN}(f) \subseteq \Phi(f)$, $\text{MAX}(f) \cap \Phi(f) = \emptyset$ *(that is, parameters of the sinusoid change at every local minima and never change at any local maxima).* □

2 Main results

In this section we present our main results on shape recovery from diameter. Theorem 2 presents a necessary and sufficient condition for two polygons to have the same diameter function. Theorem 3 shows that there are infinitely many polygons, all satisfying the conditions of Theorem 2, and all having the same diameter function.

Let d denote a valid diameter function. Two circular lists of orientations (such as $\Phi(d), MAX(d)$) are equal if they are equal after some fixed offset is added to every element of one of the lists.

From now on, maxima, minima stand for local maxima, local minima (in a diameter function), respectively. Orientations in $k(d) = \Phi(d) - MIN(d)$ are called ***kink orientations***, or more simply ***kinks***. That is, kinks are the non-minima orientations at which the parameters of the sinusoid describing d change. Kinks and minima, *i.e.* orientations in $\Phi(d) = MIN(d) \cup k(d)$, are all and the only orientations at which an edge of the polygon P is flush with (at least) one of l, h.

Let m, k, respectively denote the number of minima, kinks in $[0, \pi)$, in the diameter function d of an n-gon P. Let p be the number of pairs of parallel edges in P.

Lemma 1 $n - p = m + k$. □

Notice that the quantities on the left side n, p are the polygon's geometrical properties, while the quantities on the right m, k are properties of its diameter function. Let us refer to an n-gon having p pairs of parallel sides as an n,p-polygon. Similarly, a diameter function having m minima and k kinks is an m,k-diameter function.

Corollary 1 *If an n_1,p_1-polygon and an n_2,p_2-polygon have the same diameter function, then $n_1 - p_1 = n_2 - p_2$.* □

The *orientation of an edge e* of P is the angle modπ made by the edge with the positive x-axis. Let $ANGLES(P)$ denote an ordered circular list of orientations of the edges of P.[1] $t_P(\phi)$, $0 \leq \phi < \pi$, the partial perimeter of P restricted to orientation ϕ, is the sum of the lengths of (at most two) edges of P that have orientation ϕ. We drop the subscript when we are discussing only one polygon. Finally, $PARTIALS(P)$ denotes the circular list of the non-zero partial perimeters of P ordered in the order of increasing ϕ.

A polygon P is said to be *consistent* with a valid diameter function d *between orientations* $[\phi_a, \phi_b]$ if the diameter function of P matches d between orientations $[\phi_a, \phi_b]$. This is written as $P \sim d[\phi_a, \phi_b]$.

Throughout this paper, infinitely many stands for uncountably infinite. We will show in Theorem 2 that, in a sense, $ANGLES(P), PARTIALS(P)$ are the *maximal non-redundant (and invertible) information* of the geometry of a polygon P obtainable from its diameter function; $ANGLES(P)$ giving the orientations of the edges of P, and $PARTIALS(P)$ the partial perimeter along each orientation. However, the two lists do not completely determine P because there could be up to two edges along an orientation ϕ and the $t(\phi)$ condition is *only a constraint on the sum of the length of the two edges.* In fact, Theorem 3 shows that infinitely many polygons exist sharing the same diameter function.

Lemma 2 *Let* $0 < \phi_1 < \phi_2$ *be an adjacent triplet of orientations in* $\Phi(d)$ *of a valid diameter function d. Let* $0, \phi_1, \phi_2$ *also be an adjacent triplet in* ANGLES(P), *for some polygon P. Further, let P be consistent with d at orientations* 0 and ϕ_2. *Then, if* L, α *are the parameters of the sinusoid between* $(0, \phi_1)$, i.e. $L\cos(\alpha) = d(0), L\cos(\alpha + \phi_1) = d(\phi_1)$,

$$P \sim d[0, \phi_2] \Leftrightarrow t(\phi_1) = \frac{d(\phi_2) - L\cos(\phi_2 + \alpha)}{\sin(\phi_2 - \phi_1)}.$$

Theorem 2 *Two polygons P, Q have the same diameter function if and only if* ANGLES(P) = ANGLES(Q) *and* PARTIALS(P) = PARTIALS(Q). □

Diameter functions of parallelograms (4,2-gons) are termed *trivial.*

Theorem 3 *For every non-trivial valid diameter function d there exist infinitely many polygons having the same diameter function.*

[1] It may be noted that if d denotes the diameter function of P, then $ANGLES(P)$ is simply $\Phi(d)$ restricted to $[0, \pi)$.

Proof: Fig. 1 shows this is true for diameter functions of triangles. Towards the generalization assume that P is a polygon having diameter function d. P exists since d is valid. Let A, C be two vertices of P touching l, h at a maxima orientation. Let this maxima orientation be the zero orientation, WLOG. Let D, B be the vertices adjacent to C in P (*i.e.* DC, BC are two edges of P). Likewise, let $D*, B*$ be the vertices adjacent to A. $D*$ and D are on the same side of AC (as are $B*$ and B). $D*$ (*resp.* $B*$) could be coincident with D (*resp.* B). For example, in Fig. 1: the quadrilateral case, $D = D*, B = B*$. Let $\phi_1, \pi - \phi_2, \phi_3, \pi - \phi_4$ be the orientations of edges $CB, AB*, AD*, CD$, respectively. Without loss of generality assume $\phi_2 < \phi_4, \phi_1 < \phi_3$. The other cases (including equality) are treated similarly. See Fig. 2. F', E' are can be arbitrarily chosen on $CB, AB*$, respectively. G' is such that CG' is parallel and equal to $B*E'$. Thus we have $t(\phi_2) = |AB*| = |AE'| + |CG'|$. H' is determined similarly. It is defined so that AH' is parallel and equal to BF'. Now, $t(\phi_1) = |CB| = |CF'| + |AH'|$.

A line is drawn parallel to $AD*$ (*resp.* (DC)) through H' (*resp.* G'). Points $D*', D'$ are chosen on these two lines so that the distance between $D', D*'$ is equal to that between $D*, D$. Now the portion of the polygon P between $D*, D$ can be moved over to between $D*', D'$.

$B*', B'$ are defined in a similar manner. First a line is drawn parallel to $AD*$ (*resp.* DC) through F' (*resp.* E'). $B*', B'$ are chosen on these lines so that the distance between them equals that between $B*, B$. The portion of P between $B, B*$ can be moved over to between $B', B*'$. If this causes any problems of convexity, then take the edges $F'B'$, $E'B*'$, and those originally between $B, B*$, sort them by orientation, and arrange them between F' and E'.

Simple geometry can be applied to show that $|H'D*'|+|B'F'| = |AD*| = t(\phi_3)$ and $|G'D'| + |E'B*'| = |DC| = t(\phi_4)$. For example to show that $|H'D*'|+|B'F'| = |AD*|$, draw a line through H' parallel to $D*'D'$ intersecting $AD*$ at Z. Now note that triangle $F'B'B$ is congruent to triangle $H'ZA$ and so $|B'F'| = |AZ|$. Also note that $H'ZD*D*'$ is a parallelogram, and so $|H'D*'| = |ZD*|$.

Thus, the two polygons $P \stackrel{\text{def}}{=} A, D*, \ldots, D, C, B, \ldots, B*, A$ and $P' \stackrel{\text{def}}{=} A, H', D*', \ldots, D', G', C, F', B', \ldots, B*', E', A$ have the same diameter function by Theorem 2 since $t_P = t_{P'}$ and $\Phi_P = \Phi_{P'}$. Finally, note that there are infinitely many P' since the choices of F', E' (along a line segment) were arbitrary. □

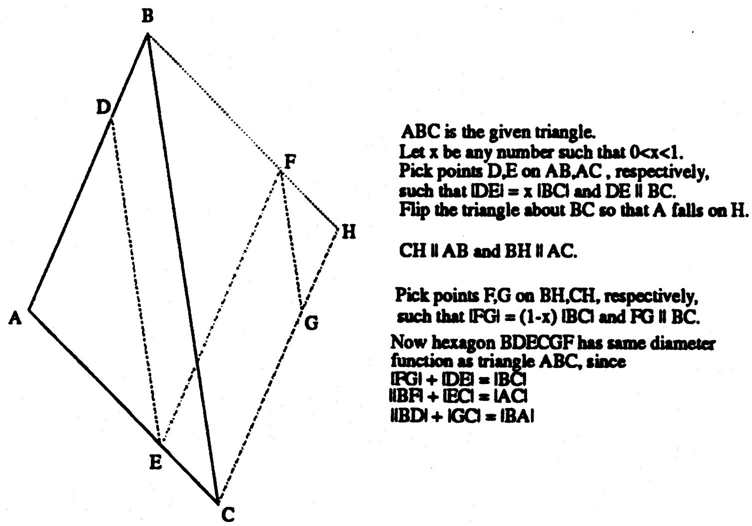
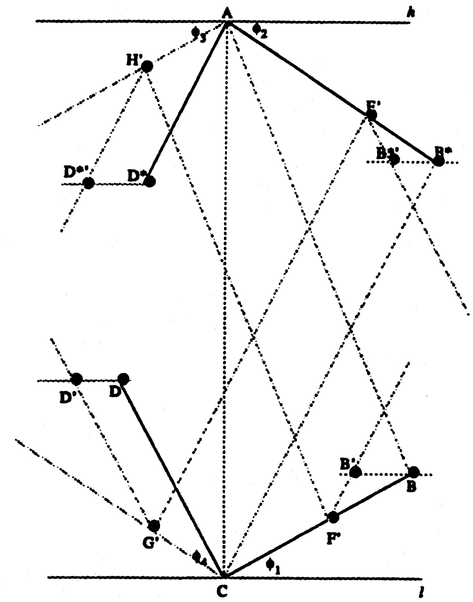


Figure 1: Infinitely many hexagons having same diameter function as given triangle/quadrilateral.

3 Minimal Polygons

Theorem 3 is a negative result for shape recovery from diameter: there exist infinitely many polygons consistent with a given measured diameter function. However, the proof of the theorem basically involved showing that a particular length $t(\phi)$ could be split, in infinitely many ways, into two segments (in the polygon), both of orientation ϕ whose lengths sum up to $t(\phi)$. Thus, most of the polygons in this infinite class would have parallel edges of varying lengths. This suggests that we might define a representative or *minimal* polygon as one without any parallel edges consistent with a given diameter function. Two obvious questions arise: does there always exist a representative polygon for a given diameter function; and if a representative polygon exists, is it always unique? We show that deciding the former question is *NP*-complete in Theorem 5 and latter question is answered in the negative in Theorem 4 by constructing a counter-example,

Formally, we define a ***minimal polygon*** to be a (convex) polygon without any parallel edges, *i.e.* an $n, 0$-polygon. Positive results for complete shape recovery from diameter of minimal polygons might be expected in the light of the easily established fact that for an m, k-diameter function there can be at most a finite number (2^{m+k-1}) of minimal polygons consistent with it. However:



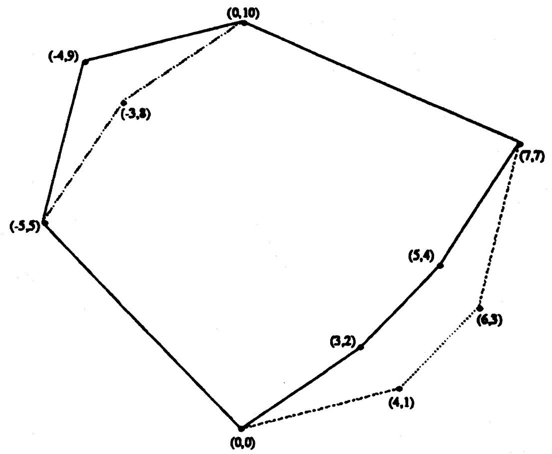
A, B*, B, C, D, D* are vertices of the original polygon P. Portions of P between B,B* and D,D* are not shown. In the new polygon P', D,D*,B,B* do not exist and are replaced by the 8 primed vertices shown. The method of construction of these eight points is shown in the proof of the theorem.

Both polygons P,P' have same perimeter along every orientation and therefore have the same diameter function.

Figure 2: Infinitely many polygons having the same diameter function as a given polygon.

Theorem 4 *Minimal polygons satisfying a given diameter function are not always unique.*

Proof: See Fig. 3. □



The two polygons, one with the shaded edges and the other with the bold edges have the following properties:
1. Both have the same set of orientations of edges.
2. Both have same total perimeter along every orientation.
3. Both have no parallel edges.
From 1,2, and Theorem 2, both have the same diameter function.

Thus, minimal polygons conistent with a given diameter function are not unique.

Figure 3: Two minimal polygons that have the same diameter function.

Now we show that deciding that whether a minimal polygon exists consistent with a given diameter function is *NP*-complete. We begin with some definitions. By ***arranging*** a set of n planar segments $S_0, \ldots, S_{n-1}$, we mean translating them in the plane so that if any two segments intersect, they do so only at their end-points. All sets in this section are multisets, *i.e.* they could contain more than one identical element. Consider the following problem.

MINIMAL_POLYGON_FROM_SEGMENTS (MPFS): Given n segments, no two of which are parallel, does there exist an arrangement of them forming a convex polygon?

Lemma 3 *MPFS is* NP-*complete.* □

Now consider this problem:

MINIMAL_POLYGON_FROM_DIAMETER _FUNCTION (MPFD): Given an m, k-diameter function d, is there a minimal polygon P consistent with d?

Theorem 5 *MPFD is* NP-*Complete.*

Proof: Let algorithm *MPFD(d)*, where d is an m, k-diameter function, return true (*resp.* false) according as whether there is (*resp.* is not) an $m+k, 0$ polygon P (fully) consistent with d.

Then we solve the *MPFS* problem using the following algorithm:

INPUT: description of n planar segments, no two of which are parallel.

OUTPUT: true/false whether or not they form a convex polygon.

1. Let *ANGLES* be a circular list of the orientations of the input segments in sorted order and let *PARTIALS* be a list of the lengths of the segments sorted according to the order in *ANGLES*.

2. By solving a system of $3n$ linear equations in $3n$ unknowns (see proof of Theorem 2), it is possible to determine whether or not there exists some (valid) diameter function d and a polygon P' such that: (i) $\Phi(d) \bmod \pi = ANGLES$, (ii) $PARTIALS(P') = PARTIALS$, and (iii) d is the diameter function of P'. The unique solution to the equations provides a construction of d if it exists.[2]

   If there does not exist such a d, return "FALSE" and exit.

3. We assume that there exists such a valid d. Now invoke *MPFD(d)*.

   Return "TRUE" if and only if *MPFD(d)* returned "true".

It is straightforward to check the correctness of the algorithm and the fact that it runs in polynomial time given that Algorithm *MPFD* does. □

4 Discussion

Since parallel jaw grippers and scanning light beams are inexpensive and widely available, we ask: Are diameter measurements sufficient to determine part shape? We show that the answer is negative by constructing an infinite class of polygonal parts with the same diameter function as any given polygon. Furthermore, deciding if there is a representative (minimal) polygon from this class is *NP*-Complete.

The good news is that we can recover some information about part shape, namely the list of edge angles and the partial perimeters at each angle. This

[2] However, remember that P' could be convex and have parallel edges.

suggests that diameter measurements could be used to recognize a part from a set of known parts. One idea is to modify the parallel jaw gripper as in [3] so that parts rotate into one of a finite number of stable orientations when grasped. Since stable diameters are not unique, one approach is to randomly grasp the part and use a Bayesian estimator to update a probability distribution on the set of parts [7] until the distribution converges on a single part.

Although random grasping appears to have good average-case performance, it's worst case performance is poor. If there are k polygonal parts, each with no more than n edges, [5] shows how to construct an $O(kn)$ time strategy for recognizing parts based on Goldberg's parts orienting algorithm. We are currently working on a method for generating optimal grasp strategies [12].

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