SHORTSIGHTED WATCHMAN ARRANGEMENT

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Abstract: The problem of arranging watchmen in a polygon with holes is considered in this paper. For watchmen with unlimited visibility, an O(N log N) approximate algorithm is presented that guarantees the number of arranged watchmen to be equal to or less than the number of reflex vertices of the environment. When watchmen are shortsighted, three approximation algorithms based on an NP-hard proof of the optimal arrangement problem and/or the above approximate algorithm for watchmen with unlimited visibility are described.

I. Introduction

Watchmen are arranged to observe and check whether or not everything is normal in known environments. If the watchman's view range in distance reaches the limits of the environment, the viewed area of a watchman forms a star polygon or a fan polygon. Arranging the minimum number of watchmen in a known environment is thus equivalent to finding the minimum number of star polygons which cover a simple polygonal region with holes. In general, this covering problem is NP-hard when Steiner points are permitted or NP-complete without Steiner points[1]. Partition of a polygon may be considered as a specific case of covering. The problem of partitioning a simple polygon with holes into a minimum number of star polygons remains NP-hard when Steiner points are permitted or NP-complete without Steiner points[2]. Aggarwal et al.[3] developed an O($N^8 log N$) approximation algorithm which obtains at most O($log N * C_{opt}$) vertex guards for a polygon that may have holes, where N is the vertex number and C_{opt} denotes the minimum number of vertex guards. O'Rourke[4] indicated that (1) for a polygon of n vertices with h holes, $\lfloor (N+2h)/3 \rfloor$ guards suffice to dominate any triangulation; (2) r guards are occasionally necessary and always sufficient to see the interior of a simple polygon of N vertices with $r \geq 1$ reflex vertices.

In Section II of this paper, an O(N log N) approximate approach for arranging watchmen with unlimited visibility is described. Based on the generalised Delaunay triangulation of a polygon with holes[5, 6], the approach is characterised by merging the triangles into convex polygons around reflex vertices, then heuristically selecting a subset of vertices as the positions for watchmen and further merging the convex polygons into star polygons. The number of the arranged watchmen is guaranteed to be equal to or less than r, the number of reflex vertices. When watchmen are shortsighted, i.e., their view range in distance is less than the limits of an environment, the task of arranging them becomes more difficult. The optimal arrangement problem is NP-hard; a proof is given in Section III. Approximation algorithms based on this proof and/or the approach for arranging watchmen with unlimited visibility are also described in Section III.

II. Arranging watchmen with unlimited visibility

Since Delaunay triangulations have a nice property, i.e., they tend to avoid giving long, thin triangles, a generalised Delaunay triangulation is considered to be a good start of a partition of the free space. After merging the generalised Delaunay triangles into convex polygons, the approach described below tries to sequentially select the positions for watchmen from those vertices whose corresponding

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star polygons formed by sets of merged convex polygons will cover the local uncovered areas as much as possible. This approach consists of the following steps:

1. Construct a generalised Delaunay triangulation of a graph in O(N log N) time, such as the example shown in Figures 1.a.
2. Check each reflex vertex and mark Delaunay edges in O(N) time.
   (a) If there are Delaunay edges within or on the inner cone of a reflex vertex (e.g., reflex vertex A in Figure 1.a), then choose and mark the shortest one (e.g., Delaunay edge AB in Figure 1.a).
   (b) If there is no Delaunay edge within or on the inner cone of a reflex vertex (e.g., reflex vertex C in Figure 1.a), then choose the two Delaunay edges which are nearest to the inner cone from the left and right sides respectively and mark them (e.g., Delaunay edges CD and CE in Figure 1.a).
3. Remove the unmarked Delaunay edges in O(N) time. The free space is now partitioned into convex components, as shown in Figure 1.b.
4. Check those reflex vertices which still have more than two Delaunay edges emanating from them. For this kind of reflex vertex, if removing any one of its two Delaunay edges chosen from Step 2(b) unites the two adjacent components into one convex component, then remove it. This results in the free space being partitioned into fewer convex components in O(N) time.
5. Label each of the convex components so formed with a pair of numbers: the number of reflex vertices located on the convex component and the number of the Delaunay edges located on the convex component in O(N) time (See Figure 1.b).
6. Label each vertex with a pair of numbers, which are the sums of the label numbers of the convex components around the vertex, in O(N) time (See Figure 1.b).
7. Using the following greedy method, repeatedly select candidate positions for watchmen from vertices on the basis of their vertex label values in O(N log N) time:
   (a) Construct a heap to store vertices, first according to their first label values and then according to their second values, when the first values are same. If both values are same, a reflex vertex is first. The vertex with the biggest label value is set at the root. The information on pointers pointing to the adjacent convex components of a vertex is stored along with the vertex. This requires O(N) time.
   (b) Check the vertex of the root in the heap; if none of its adjacent convex components have been marked, then select it as a candidate position, mark those convex components with the vertex identifier and delete this vertex from the heap. Otherwise, change the label value of the vertex into the difference between its original value and the values of those marked adjacent convex components.
      i. If the new value is (0, 0), then delete this vertex from the heap.
      ii. If the new value is equal to its previous value, then select this vertex as a candidate position, mark those unmarked adjacent convex components with the vertex identifier and delete this vertex from the heap.
      iii. Otherwise rearrange the heap according to the new value. The original value of the vertex is stored along with the new value of the vertex.
   (c) Repeat Substep 7(b) until no vertices are in the heap. The selected candidate positions for watchmen along with the convex components marked by their identifiers form a star polygon partition as shown in Figure 1.c.
8. For those last selected vertices which marked only one convex component, it is required to check the positions for watchmen whose corresponding star polygons are adjacent to the marked-out convex component, i.e., their star polygons share a remaining Delaunay edge with the marked-out convex

component. If the convex component can be partitioned or covered by the extended star polygons viewed from these adjacent positions for watchmen, then the candidate position associated with the convex component should be deleted.

This kind of vertex has a low number of Delaunay edges located on its marked convex component. If the number of Delaunay edges located on the marked convex component is limited to a constant value – a heuristic parameter, then the check for those vertices can be done in O(N) time.

9. A further check is required to be done on those candidate positions for watchmen located at convex vertices.

   A reflex vertex is dominated by a convex vertex, iff (1) the reflex vertex is located on the star polygon viewed from the convex vertex (a candidate position selected in Step 7); (2) after Step 4, the reflex vertex has a remaining Delaunay edge which connects the reflex vertex with the convex vertex; and (3) the two convex components at the two sides of the above Delaunay edge are marked by the identifier for the convex vertex in Step 7. After a convex vertex has been selected as a candidate position, if all its dominated reflex vertices are then selected as candidate positions for watchmen in Step 7, then the star polygons formed by merging the convex components round these reflex vertices can cover the star polygon marked by the identifier for the convex vertex. In this case, we eliminate the candidate position at the convex vertex; otherwise we still keep it. Since the number of Delaunay edges remaining after Step 4 is equal to or less than 2r, this check can be done in O(N) time.

The final remaining candidate positions for watchmen are the selected positions for watchmen. The result for the example graph is shown in Figure 1.c. According to Euler's theorem, the number of the selected positions for watchmen is equal to R-h+1, where h is the number of holes and R is the number of the last remaining Delaunay edges, as shown in Figure 1.c. The upper bound of the number of watchmen can be proved as r, the number of reflex vertices. In accordance with O'Rourke's conclusion[4], the upper bound of the approach is tight for general situations.

III. Arranging shortsighted watchmen

When a watchman's view range in distance is less than the limits of its environment, a viewed area of the watchman forms a star disk or a fan disk. A Star Disk is defined as a plane graph which is an intersection region of a disk and a star polygon where the center of the disk is located in the kernel of the star polygon. The center of the disk is the center of the formed star disk. If the intersecting disk has diameter D, then the formed star disk also has diameter D. If D is large enough for the disk to include the star polygon, the formed star disk degenerates into the star polygon. A fan disk is a star disk whose center coincides with one of its vertices. We designate the problem of finding the minimum number of star disks with diameter D which cover a simple polygonal region with holes by the name of the Minimum Star Disk Cover problem. In general, it is NP-hard. A proof is given below:

1. Construct a generalised Delaunay triangulation of a simple polygon with holes.

2. For each triangle which cannot be covered by a disk with diameter D, partition the triangle into two by bisecting its biggest angle. Repeat this process for each triangle, until every triangle can be covered by a disk with diameter D.

3. Let S be the set of triangles formed at Step 2. If several triangles can be covered by the same star disk with diameter D whose center is located at a node, which is the circumcenter of an acute triangle or the mid-point of the longest edge of a right or obtuse triangle, then they form a subset of S. Assume that C is the collection of all the subsets, then the original optimization problem can be restricted to the Minimum Cover problem: For collection C of subsets of a finite set S, positive integer $K \leq |C|$, does C contain a cover for S of size K or less? This problem is known as an NP-complete problem even if all $c \in C$ have $|c| \leq 3$. It is solvable in polynomial time by matching techniques only if all $c \in C$ have $|c| \leq 2$.

Since Steps 1 to 3 require only polynominal time and a star disk may cover more than 2 triangles, we can conclude that the original optimization problem is NP-hard.

Three approximation algorithms of arranging the positions for watchmen are listed below. Their application depends on the relative size between the watchman's view range in distance D and the dimensions of the environment. Clearly, their solutions depend on the shape of the simple polygon with holes and the diameter D of the covering star disks.

When D can be compared with the minimum length of the generalised Delaunay edges of a polygon with holes, a modified version of the above NP-hardness proof can be used as an approximation algorithm. It consists of:

1. Construct a generalised Delaunay triangulation of a simple polygon with holes in O(N log N) time[5, 6].
2. In the triangulation, assume that $\triangle$ABC has its edges "a", "b" and "c", "a" is its longest edge and "b" is its shortest edge. If D < $a * \sec(B/2)$, then partition the triangle into two by bisecting $\angle A$. Repeat this process for each triangle, until D $\geq a * \sec(B/2)$, i.e., every triangle can be covered by a disk with diameter D whose center O is located at the intersection point of the bisector of $\angle B$ and the line which is perpendicular to "a" and passes through the mid-point of "a". The finer triangulation for the example graph is shown in Figure 2.a.

   Let D_s be the diameter of the smallest disk covering $\triangle$ABC and D_o be the diameter of the smallest disk covering $\triangle$ABC, whose center is located at O. In all cases, D_o less than 1.083 * D_s.
3. Construct a dual graph for the finer triangulation (see Figure 2.b). This dual graph consists of nodes and arcs which connect every two nodes whose corresponding triangles share a common edge. A node in $\triangle$ABC is located at the intersection points of the bisector of the smallest angle B and the line which is perpendicular to the longest edge "a" and passes through the mid-point of "a". A node may have 1 to 3 connecting arcs, therefore it has a degree between 1 and 3.
4. The following greedy method is used to find a star disk partition for the free area:
   (a) Select a node with degree 3 in the dual graph as a watchman's position in O(N) time.
   (b) From this node, along the unexplored arcs, test the vertices of the unexplored adjacent triangles and find the adjacent triangles which can be covered by the star disk with diameter D whose center is located at the node. Merge the covered triangles into a star polygon associated with the node. Repeat this process, until there are no more adjacent triangles which can be merged in. Assuming the number of merged triangles for position i is M_i, this step requires O(M_i log M_i) time.
   (c) Along an unexplored arc of the dual graph, consider the next unexplored node:
      i. If the next unexplored node has degree 3, then select the node as a watchman's position (e.g, position 4 in Figure 2.b) and go to Step 4.b.
      ii. If the next unexplored node has degree 2, check a star disk with diameter D whose center is located at the unexplored node after the next node, to determine whether it can cover the next triangle.
         A. If it can cover, select it as a watchman's position (e.g, position 3 in Figure 2.b) and go to Step 4.b.
         B. If it cannot cover, then check a star disk with diameter D whose center is located at the mid-point between the next unexplored node and the node after the next, and determine whether it can cover both the next triangle and the triangle after the next.
         (1) If it can cover, select it as a watchman's position (e.g, position 6 in Figure 2.b) and go to Step 4.b.
         (2) If it cannot cover, select the node of the next triangle as a watchman's position (e.g, position 5 in Figure 2.b) and go to Step 4.c.

iii. If the next unexplored node has degree 1, select it as a position.

This step requires constant time.

(d) Repeat the above step (c) until all triangles are covered.

The whole process requires O(N log N) time, since $\sum M_i$ is O(N). This algorithm in fact partitions the free region into star polygons, each of which is a degenerated star disk with diameter D. Figure 2.b shows the result of the example graph by this approach.

When D is quite large, the result of the approach for watchmen with unlimited visibility shows that only few decomposed star polygons cannot be covered by the star disks with diameter D whose centers locate on the selected positions for watchmen. In this case we may modify the result of the approach as follows:
For each star polygon which cannot be covered by one star disk with diameter D, consider its generalized Delaunay triangulation and further use the first approximation algorithm described in this section to find the positions for watchmen.

When D is quite small compared with the dimensions of the environment, we partition the simple polygon (with or without holes) by a grid, in which each cell can be covered by a disk with diameter D. The free area in a cell can be classified as:

1. a star polygon with the cell center in its kernel,
2. a simple polygon,
3. a simple polygon with holes,
4. several disconnected components, each of which may be a simple polygon with holes.

For the first type of cells, a watchman's position can be placed on the center of a cell. For the second and third types of cells or a component of the forth type of cells, the approach given in Section II can now be applied.

IV. Conclusion

We have presented an O(N log N) approximate approach for arranging watchmen with unlimited visibility that guarantees the number of arranged watchmen to be equal to or less than r, the number of reflex vertices of the environment figure. When watchmen are shortsighted, we have proved that the optimal watchman arrangement problem is NP-hard. Three approximation algorithms based on this proof and/or the approach for arranging watchmen with unlimited visibility are described. Their application depends on the relative sise between the watchman's view range in distance D and the dimensions of the environment.

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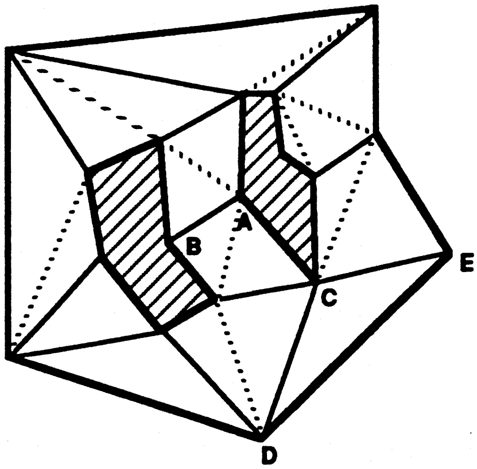
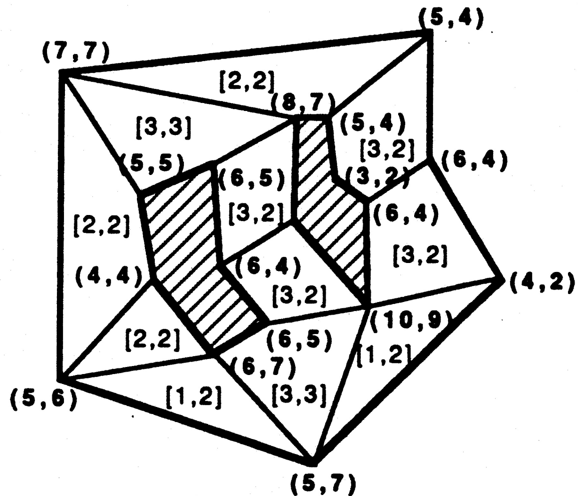
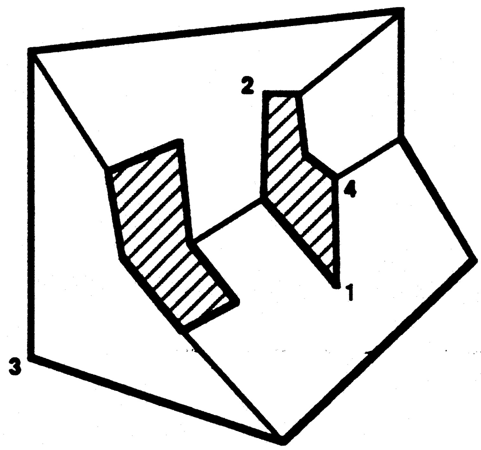


(a) The marked Delaunay edges shown by thin solid lines.

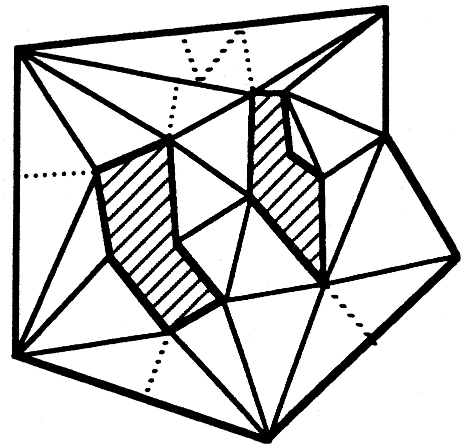


(b) The labelled convex components and reflex vertices.

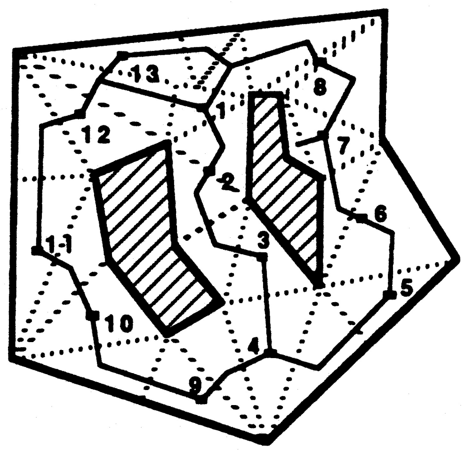


(c) The selected positions and the decomposed polygons.

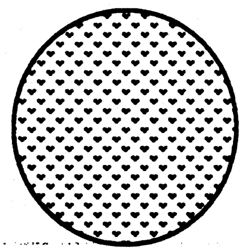
Figure 1: The arrangement for watchmen with unlimited visibility for an example graph.



(a) The fined triangulation.



(b) The dual graph and the selected positions by a greedy method.



(c) The view range of a shortsighted watchman.

Figure 2: The arrangement for shortsighted watchmen.