Efficiently Updating Constrained Delaunay Triangulations

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Abstract
The *Constrained Delaunay Triangulation* of a set of obstacle line segments in the plane is the Delaunay triangulation of the endpoint set of these obstacles with the restriction that the edge set of the triangulation contains all these obstacles. In this paper, an optimal $\Theta(\log n + k)$ algorithm for inserting an obstacle line segment or deleting an obstacle edge in the constrained Delaunay triangulation of a set of n obstacle line segments in the plane is presented, where k is the number of Delaunay edges deleted and added in the triangulation during the updates. The above result is based on a linear-time algorithm for finding the constrained Delaunay triangulation of a specific polygon, called the *Delaunay monotone* polygon.

1 Introduction

Delaunay triangulation and its dual, Voronoi diagram, are two important data structures in computational geometry. The two structures for a set of points (called sites) have been extensively studied [3, pp.198-218, 12]. Motivated by geographical interpolation problems, Lee and Lin [7] first investigated Delaunay triangulation in the presence of obstacles. They proposed an $O(n^2)$ algorithm for finding the constrained (they called *generalized*) Delaunay triangulation of a set of n line segments as well as an $O(n \log n)$ algorithm for the triangulation of a simple polygon, where the endpoints of the line segments and the vertices of the polygon are regarded as sites and the line segments and the edges are regarded as obstacles. Later, the time bound for solving the problem in the first case was reduced to $\Theta(n \log n)$ [4,13,14]. However, whether or not the problem in the second case can be solved in $o(n \log n)$ time is not known [1]. This outstanding open problem for the case of a convex polygon has been solved by Aggarwal et al. [2], but the problem for the case of a general simple polygon still remains open. Recently, a linear-time algorithm for the problem of a monotone histogram has been presented by Djidjev and Lingas [5], and thus the problem of a general simple polygon can be solved in $O(n \log r)$ time for $r < n$. In this paper, we show a linear-time algorithm for the problem in the case of a special simple polygon, called the Delaunay monotone polygon. Moreover, the duality of constrained Delaunay triangulation and constrained Voronoi diagram of a set of line segments is studied in [6].

For updating (that is, inserting or deleting a site in) the Voronoi diagram of a set of n sites, Aggarwal et al. [p.601, 2] proposed an optimal $\Theta(\log n + k)$ method for deleting a site in the diagram, where $\log n$ is the time for point location and k is the number of Voronoi edges deleted and added during the update process. Since inserting a site to the diagram takes obvious $\Theta(\log n + k)$ time, the diagram can be updated in $\Theta(\log n + k)$ time. By the duality, the corresponding Delaunay

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triangulation can be updated in the same time bound. However, whether or not constrained Delaunay triangulations and constrained (or called *bounded*) Voronoi diagrams can be updated efficiently is not known. Directly using existing algorithms [4,13,14] for updates is inefficient because these algorithms are required to rebuild the entire triangulation. In this paper, we present an optimal $\Theta(\log n + k)$ time update algorithm.

For simplicity, we use CDT and $CVor$ to denote the terms: constrained Delaunay triangulation and constrained Voronoi diagram respectively.

2 Finding the CDT of a special polygon

In Definitions 1 to 4, we shall define what is constrained Delaunay triangulation and its dual, extended constrained Voronoi diagram. Let L denote a set of non-intersecting line segments (except possibly at their endpoints) representing obstacles. Let S denote the endpoint set of L.

Definition 1: The distance of $s \epsilon S$ and $x \epsilon R^2$ in the presence of obstacles is determined by

$$d_L(x,s) = \begin{cases} d(x,s) & \text{if } x \text{ and } s \text{ are visible from each other,} \\ \infty & \text{otherwise} \end{cases}$$

Definition 2: The CVor of L, denoted by $CVor(L)$, is a set of Voronoi cells $\{V(s) \mid s \epsilon S\}$, where $V(s) = \{x \epsilon R^2 \mid d_L(x,s) \leq d_L(x, s')$ and $d_L(x,s) \neq \infty$, for all $s' \in S$, $s \neq s'$ $\}$.

The **boundary** of a Voronoi cell $V(s)$ is the closure of $V(s)$. A **Voronoi edge** is a maximal straight line segment on the boundary of a Voronoi cell. A **Voronoi vertex** is an endpoint of a Voronoi edge. The $CVor$ of a simple polygon is a special case of $CVor(L)$ that L is a simple polygon.

Definition 3: Let P be a simple polygon, where the interior of P is on the right hand side of the directed boundary. Let every edge e of the boundary of P be attached by a **sheet** SH_e, where SH_e is a half-plane on the left hand side of the line extending e and SH_e is on the top of R^2. Let E denote the space formed by these sheets and P such that the sheets and the interior of P are pairwise disjoint. The visibility on E is defined as follows: Two points x and y are *visible* from each other if for some e and e' of P, (1) both $x, y \,\epsilon SH_e$ or (2) $x \epsilon SH_e$ and $y \epsilon SH_{e'}$ and $\overline{xy}$ crosses exactly e and e' or (3) $x \epsilon SH_e$ and $y \epsilon P$, and $\overline{xy}$ crosses exactly e or (4) $x, y \epsilon P$ and $\overline{xy}$ lies entirely on P. The distance of two points x and y is defined as the Euclidean distance $d(x,y)$ if x is visible to y, as infinite, otherwise. The **Extended CVor** of P on E, denoted by $ECVor(P)$, is defined in the same manner as $CVor(L)$ on R^2 with the above distance measurement.

Definition 4: The **CDT** of L, denoted by $CDT(L)$, is a triangulation of S such that its edge set contains L, and the circumcircle of any triangle, say $\Delta ss's''$, does not contain any element of $S - \{s, s', s''\}$ visible to all s, s', and s''. The CDT of a simple polygon is defined similarly.

It has been shown that the straight line dual of $CDT(P)$ is $ECVor(P)$, and one can be obtained from the other in linear-time [6,13].

In the following, we shall deal with a special polygon, called the Delaunay monotone polygon. The corresponding definitions and lemmas are presented.

Definition 5: A polygon P is called a **Delaunay monotone polygon** iff there exists a straight line (called the **cutting line**) which crosses every internal Delaunay edge of $CDT(P)$.

An example of a Delaunay monotone polygon P and its $CDT(P)$ are shown in Fig.2.1a. A Delaunay monotone polygon may not be simple since an edge (or a vertex) of the polygon may appear on its boundary twice. But, a simple polygon can be obtained by the standard perturbation method in at most linear-time since its edges do not cross. A Delaunay monotone polygon may not be monotone w.r.t. their xy-coordinates. An example of $ECVor(P')$ is shown in Fig. 2.1b, where P' is the remaining subpolygon of P truncated by a cutting line l of P. If the cutting line does not pass through two vertices of P as shown in the figure, where vertices p_1, p_2, p_3, and p_9 are cut off so that edges $\overline{p_3p_4}$ and $\overline{p_9p_8}$ are removed too, then we imagine there exist a point $p\epsilon l$ at $+x_\infty$ and a point $p'\epsilon l$ at $-x_\infty$ so that chain $\mathbf{b}=(p_4,p_5,p_6,p_7,p_{6'},p_8)$ and p and p' determine a simple polygon P'. Then, $ECVor(P')$ and $CDT(P')$ are defined. For simplicity, we use $ECVor(\mathbf{b})$ and $ECVor(P')$ (respectively,$CDT(\mathbf{b})$ and $CDT(P')$) interchangeably.

In the following, we shall show that $CDT(P)$ for a Delaunay monotone polygon P can be found in linear-time, and $CDT(P')$ can also be found in linear-time.

Lemma 2.1: Let P be a Delaunay monotone polygon. $CDT(P)$ can be found in $O(|P|)$ time.

Proof: Omitted. □

Fact 2.1: (Aggarwal et al. [p.602, 2]) Given a set S of n sites in the upper halfplane, sorted by their x-coordinates, the Voronoi diagram on the lower halfplane as well as the sequence of associated sites can be found in $O(n)$ time.

We will use the following notations in this subsection. Let P denote a Delaunay monotone polygon and l be a cutting line of P; P' and P'' denote the two subpolygons divided by l; (we shall consider P' only since the cases for P' and P'' are symmetric.) Let $Q_{P'}$ denote the sequence of those vertices in P' affecting the Voronoi diagram on sheet SH_l. Let **b** or **d** (or the letters with primes) denote a subchain of P'. Then, Q_b or Q_d w.r.t. a properly chosen line can be defined similarly. It is easy to see that the part of $ECVor(P')$ on SH_l is the part of $ECVor(Q_{P'})$ on SH_l by their definitions.

We shall show that $Q_{P'}$ as well as the part of $ECVor(Q_{P'})$ on SH_l can be found in $O(|P'|)$ time using Fact 2.1, then by the duality, $CDT(Q_{P'})$ can be found in $O(|P'|)$ time (Lemma 2.2).

Lemma 2.2: Let $Q_{P'}$ denote the sequence of the vertices in P' affecting $ECVor(P')$ on SH_l. $Q_{P'}$ as well as $CDT(Q_{P'})$ can be determined in $O(|P'|)$ time.

Proof: Omitted. □

Definition 6: A **bag** is a subchain $\mathbf{b}=(p_h,...,p_t)$ of P' such that p_h and p_t (which are called *head* and *tail*, respectively) are two consecutive vertices of $Q_{P'}$ and there exists at least a vertex of $\mathbf{b}-\{p_h,p_t\}$ which does not belong to $Q_{P'}$.

Hence, the vertices of $Q_{P'}$ divide the boundary of P' into a sequence of bags, denoted by B. If B is empty, then $CDT(Q_{P'})=CDT(P')$ obviously. Otherwise, it is necessary to show how to construct

$CDT(B)$. To do so, we first prove that the interior vertices of any bag **b** of B do not affect the CDT(**b'**) of other bag **b'** of B, and they also do not affect $CDT(Q_{P'})$ (Lemma 2.4). Thus, CDT(**b**) can be constructed independently of the rest of P'. We then show that CDT(**b**) can be found in linear-time (Lemma 2.6). Finally, we combine CDT(**b**) for all **b**ϵB and $CDT(Q_{P'})$ to $CDT(P')$ (by Lemma 2.4 and Lemma 2.5).

Definition 7 and Lemma 2.3 are used in the proofs of the other lemmas.

Definition 7: Let **b**$=(p_h, ..., p_t)$ be a bag (w.r.t. l). The **forbidden circle** of **b**, denoted by F_b, is a circle centered at a point $o \epsilon l$ with radius $\overline{op_h}$ such that $\overline{op_h} = \overline{op_t}$.

Lemma 2.3: Let **b**$=(p_h, ..., p_t)$ be a bag (w.r.t. l) of P'. Let $V' = (v_b, ..., v_k)$ be the sequence of Voronoi vertices of $ECVor(P)$, each of which is determined by three vertices at least one in **b** and one in P'', where P' and P'' are two subpolygons of P divided by l. In particular, v_b is determined by p_h, p_{h+1}, and a $p \epsilon P''$ and v_k is determined by p_{t-1}, p_t, and a $p' \epsilon P''$. Then, (1) no vertex of **b**$-\{p_h, p_t\}$ can lie inside or on the forbidden circle F_b, (2) any $v \epsilon V'$ must lie on the area bounded by $\overline{p_h o}$, $\overline{op_t}$, and bag **b**, and (3) all the Voronoi vertices of $ECVor$(**b**) not belonging to $ECVor(P)$ lie below V'.

Proof: Omitted. □

Lemma 2.4: Let **b** and **b'** be two bags of B. Let p and q be an interior vertex of **b** and **b'**, respectively. Then, $\overline{pq}$ is not an edge of $CDT(P')$.

Proof: Omitted. □

Lemma 2.4 implies that $CDT(Q_{P'})$ and CDT(**b**) for any **b**ϵB can be found independently and $CDT(P')$ consists of these triangulations. The following Lemma is an extension of that for the standard Voronoi diagrams.

Lemma 2.5: Let L_1 and L_2 be two linearly separable sets of obstacle line segments. Then, given $ECVor(L_1)$ and $ECVor(L_2)$, $ECVor(L_1 \cup L_2)$ can be found in $O(|L_1 \cup L_2|)$ time.

Lemma 2.6: CDT(**b**) for any **b**ϵB can be found in O(| **b** |) time.

Proof: By applying Lemma 2.2 to **b** (w.r.t. l', which will be defined), we can identify a sequence of vertices from **b**, denoted by Q_b, affecting the part of $ECVor$(**b**) on sheet $SH_{l'}$, and identify a set of smaller bags D. We then apply Lemma 2.2 to each of these bags in D again (w.r.t. a properly chosen line). It can be shown that no smaller bag will remain in any bag **d** of D. Consequently, this lemma will follow Lemma 2.4 and Lemma 2.5. We omit the details here. but refer to Fig.2.6. □

Theorem 2.1: Let P' and P'' be the two subpolygons of a Delaunay monotone polygon P divided by its cutting line. $CDT(P')$ and $CDT(P'')$ can be found in $O(|P|)$ time.

Proof: By Lemma 2.4, Lemma 2.5, and Lemma 2.6. □

3 The algorithm for updating $CDT(L)$

Let $CDT(L)$ be the constrained Delaunay triangulation of L, where L is a set of nonintersecting line segments whose endpoints are regarded as sites and whose line segments are regarded as obstacles. We consider the insertions and deletions of the endpoints of an obstacle and its open line segment in

$CDT(L)$ separately. To do so, we maintain a copy of the dual, $ECVor(L)$, of $CDT(L)$, and operate the updates on $ECVor(L)$ and then obtain $CDT(L)$ from $ECVor(L)$.

(a) The insertion and deletion of a site in $ECVor(L)$.

The method is similar to this for inserting and deleting a site p into a standard Voronoi diagram of S. We omitted the details here.

(b) The insertion and deletion of an open line segment in $ECVor(L)$.

Lemma 3.1: Let l^o denote the open line segment of l and P_l denote the corresponding monotone Delaunay polygon crossed by l. The difference between $CDT(L)$ and $CDT(L \cup \{l^o\})$ or between $CDT(L)$ and $CDT(L - \{l^o\})$ is $CDT(P_l)$.

Proof : Omitted. □

For deleting the open line segment l^o of an element l of L from $CDT(L)$, we identify the two sheets attached to l in space E. Then superimpose the parts of $ECVor(L)$ on the sheets to that on the main plane and merge them by a merge-process stated in Appendix C. Clearly, it takes $O(k)$ time, where k is the number of Voronoi edges being deleted or being added during the merge process. By the duality, we obtain $CDT(L - \{l^o\})$ in $O(k)$ time by updating the proper Delaunay edges.

For inserting the open line segment l^o of l to $CDT(L)$, we traverse l and find the Delaunay monotone polygon w.r.t. l by counting the Delaunay triangles crossed by l. Let P_l be the polygon and P_l' and P_l'' be its two subpolygons. Then, $CDT(P_l')$ and $CDT(P_l'')$ can be found in $O(|P_l|)$ time by Theorem 2.1

Theorem 3.1: $CDT(L)$ can be updated in $O(\log n + k)$ time, where $k = |P_l|$.

4 Concluding remarks

We presented an efficient algorithm for updating constrained Delaunay triangulations. The algorithm for finding the constrained Delaunay triangulation of a Delaunay monotone polygon can also be applied to finding the greedy triangulation of a set of n points in $O(n^2)$ time. We omit the details here.

5 References

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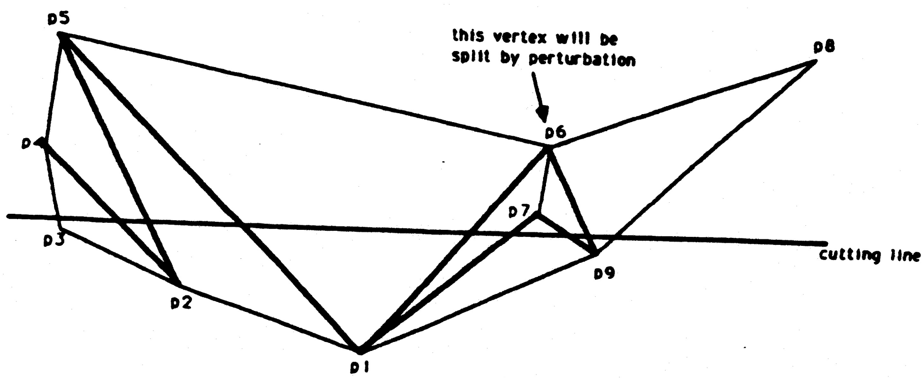
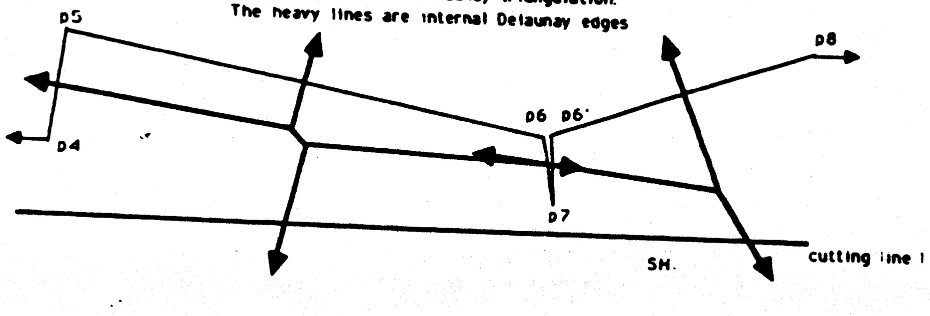


(a) An example of a Delaunay monotone polygon and its constrained Delaunay triangulation. The heavy lines are internal Delaunay edges



(b) An example of ECVor(P'), where P' is formed by p4,p5,p6,p7,p6',p8, and the two points respectively at the + and - infinite of l

Fig. 2.1

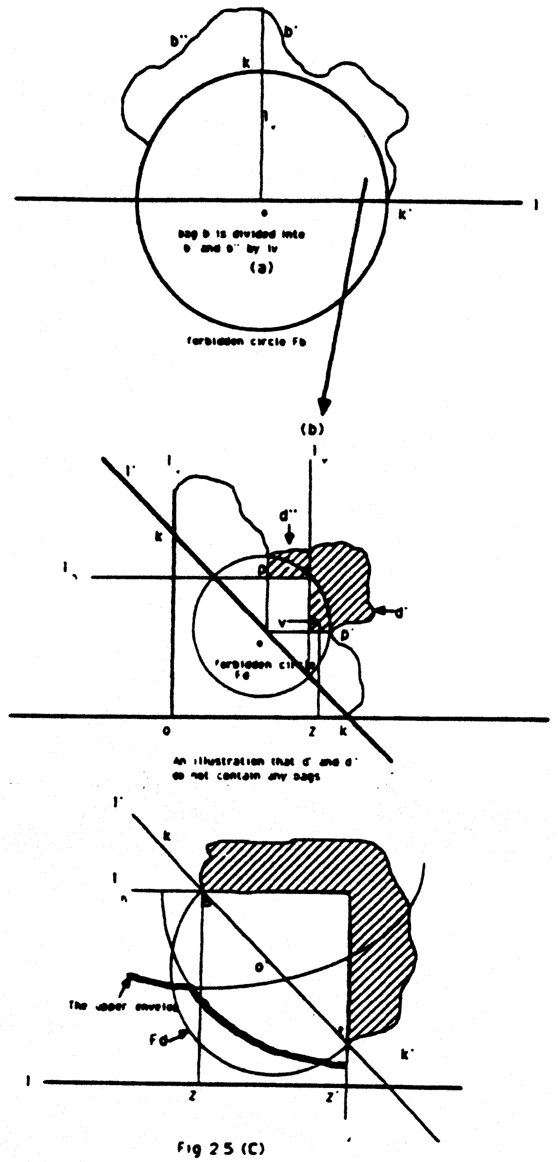


Fig 2.5 (C)