On Polygons Enclosing Point Sets

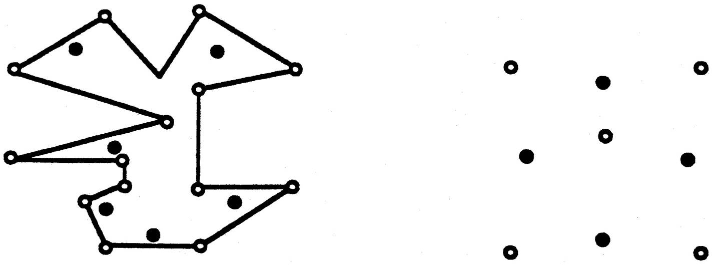
J Czyzowicz, *Université du Québec à Hull, Canada*
F. Hurtado, *Universidad Politécnica de Cataluña, Spain*
J. Urrutia, *University of Ottawa, Canada*
N. Zaguia, *University of Ottawa, Canada*

Abstract

Let P and Q be collections of of n and m points respectively, on the plane in general position. We say that P encloses Q if there is a closed simple polygon C with vertex set P such that all the elements of Q lie in the interior of C. Clearly if the elements of Q are not contained in the convex hull of P, Q cannot be enclosed by P. In this paper we prove that if Q is contained in the convex hull of P then P encloses at least half of the points of Q, and we will give examples to show that this bound is asymptotically tight. We also prove that if the polygon defined by the convex hull of P has at least m (2 log(m) +1) vertices then P encloses all the m points of Q.

1. Introduction

Let P_n and Q_m be collections of n and m points on the plane such that Q_m is contained in the interior of the convex hull, Conv(P_n), of P_n. We say that P_n encloses Q_m if there is a simple polygon C whose vertex set is exactly P_n such that Q_m is contained in the interior of C (See Figure 1(a).) For example if Conv(P_n)=k < n a collection Q_k not enclosable by P_n can be obtained by placing for each edge e of Conv(P_n) a point p_e in the interior of Conv(P_n) at a distance ε of the mid point of e, where ε small enough (see Figure 1(b).)



The set of solid points is enclosable by the set of clear points.
(a)

The set of solid points is not enclosable by the set of clear points.
(b)

Figure 1.

It is clear from the example in Figure 1(b) that the condition of Q_m being contained in the interior of Conv(P_n) is not sufficient to guarantee that it is enclosable by P_n. It is thus natural to ask the following: *Given any two collections* P_n *and* Q_m *of points such that* Conv(P_n)$\supseteq Q_m$*, Is there is a large subset* H *of* Q_m *that is enclosable by* P_n?

Throughout this paper we will denote by PConv(P_n) the polygon defined by the convex hull of P_n, Conv(P_n), and a polygon whose vertex set is P_n will be called a P_n-polygon.

Theorem 1: Given any two collections of points P_n and Q_m such that Conv(P_n)$\supseteq Q_m$ there is a P_n-polygon that encloses at least half of the points of Q_m.

Our result follows immediately from the next lemma which is interesting on its own:

Lemma 1: Let P_n be any collection of points. Then there are two P_n- polygons whose union covers entirely Conv(P_n).

Proof: Let e be an edge of Conv(P_n) with end points u and v and let p_e be the mid-point of e. Sort the points of P_n in the clockwise direction with respect to p_e and relabel them $u=p_1,\ldots p_n=v$ accordingly (See figure 2(a).) Let S be the subsequence of $u=p_1,\ldots p_n=v$ defined by $[P_n\text{-Conv}(P_n)]\cup\{u,v\}$. Let Φ_1 be the P_n-polygon $p_1,\ldots,p_n,p_1$ (See figure 2(a).)

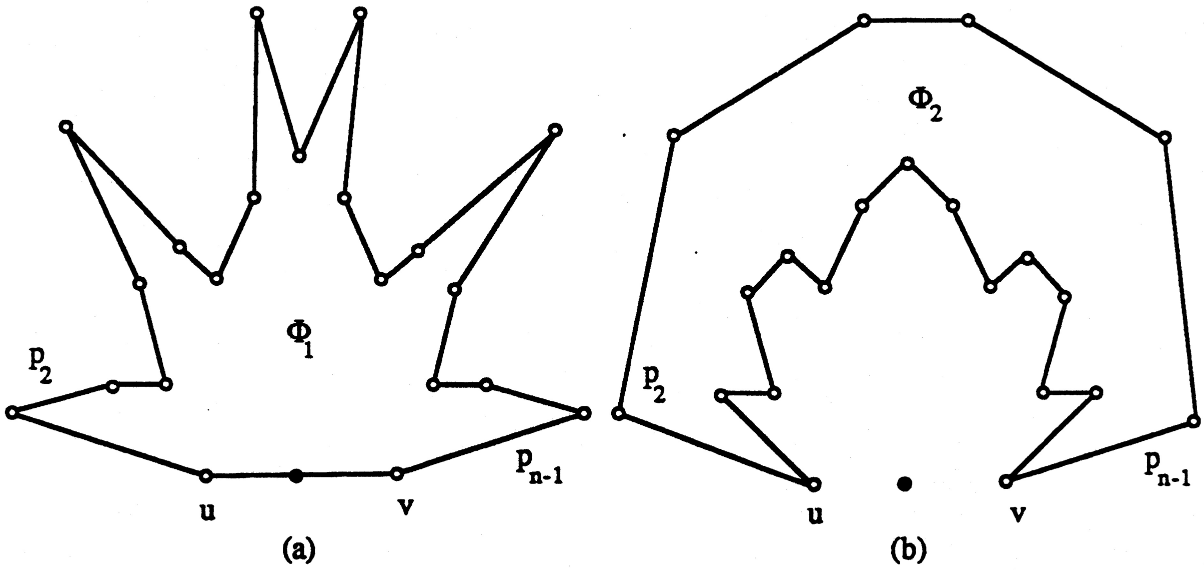


Figure 2.

We define a second P_n-polygon Φ_2 as follows: The boundary of Φ_2 consists of the union of two polygonals the first of which is Conv(P_n)-e and the second is the polygonal defined by the subsequence S of $u=p_1,...p_n=v$ (See Figure 2(b).) It is now clear that the union of Φ_1 and Φ_2 covers Conv(P_n).

Proof of Theorem 1: Since the union of Φ_1 and Φ_2 covers Conv(P_n) and Q_m is contained in Conv(P_n) then either Φ_1 or Φ_2 contains at least half of the elements of Q_m.

Next we prove that the bound in Theorem 1 is asymptotically tight. To prove this, consider a set P_n with n points such that PConv(P_n) is a triangle. Around each point $p \in P_n$ in the interior of Conv(P_n) draw a circle C_p with radius ε, ε small enough. Place r points uniformly distributed on C_p, where r is large enough. For each vertex p_i of PConv(P_n), let α_i be the internal angle of PConv(P_n) at p_i; Place $(|\alpha_i|/2\pi)r$ points uniformly at distance ε from p_i within α_i for i=1,2,3. Let Q be the set of points placed on the small circles around the points of P_n. Clearly Q contains (n-3)r + r=(n-2)r points. Since the sum of the internal angles of any P_n-polygon is (n-2)π, then it encloses at most [(n-2)r] /2 + n points. It follows that any such polygon contains at most $|Q|/2 + n$ points. As $r \rightarrow \infty$ this converges to $|Q|/2$.

It is natural to ask the following:

Can we obtain some general conditions that guarantee that a point set Q_m is enclosable by a given point set P_n? Is there a condition on m which guarantees that Q_m it is enclosable by P_n?

The answers to these questions seem to be linked to the size of Conv(P).

Theorem 2: If Q_m is contained in the interior of $Conv(P_n)$ and $m (2 \log(m) + 1) \leq |Conv(P)|$ then Q_m is always enclosable by P.

Here are some preliminary results needed to prove our result. The following lemma is given without a proof:

Lemma 2: Let P_s be a point set and let x, y and z be three consecutive vertices of $PConv(P_s)$ and q be a point in the interior of $Conv(P_s)$. Then there is a P_s-polygon Φ that encloses q such that all edges of $PConv(P_s)$ are in Φ, except possibly the edges joining x to y and y to z (See Figure 3.)

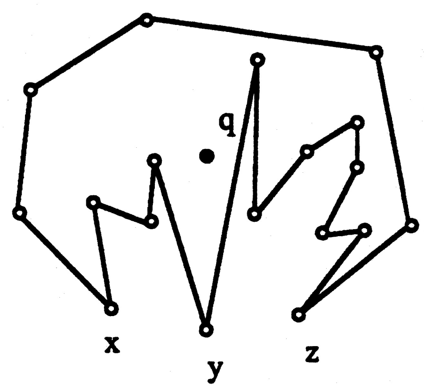


Figure 3.

We recall the following well known result:

Theorem 3 (Borsuk-Ulam): Given any two collections of points P and Q on the plane, there is a line that simultaneously bisects them both.

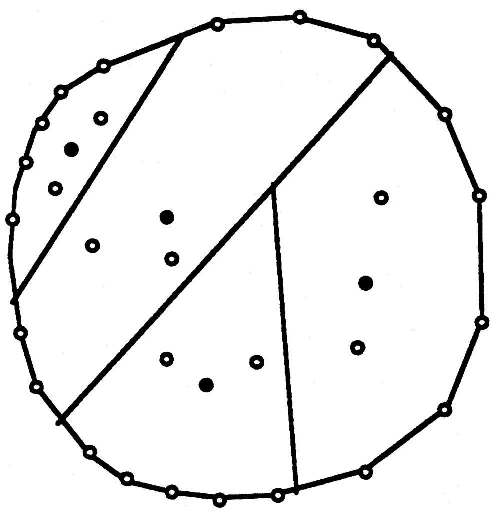
Sketch of the proof of Theorem 2: Asumme that $|Conv(P_n)| \geq m(2 \log(m)+1)$ and that $m = 2^i$ is a power of 2. (Other forms for m are handled similarly.) By theorem 3, we can simultaneously bisect the vertex set of $PConv(P_n)$ and Q_m with a line L. Iterate this process i times until Q_m has been splitted into singletons and the vertives of

PConv(P_n) have been divided into 2^i subsets, each with size at least $|\mathrm{Conv}(P_n)|/2^i$ that is into subsets of size at least 2log(m)+1(See Figure 4.) Clearly for each point q of Q_m we define in a "natural way" a polygon Φ_q that contains it and whose vertices are:

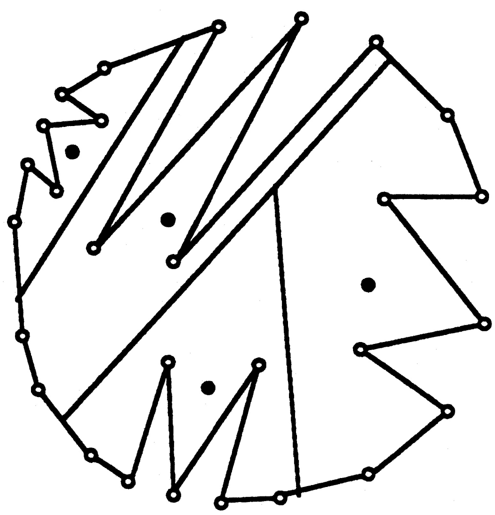
a) either intersection points of the lines used to split Q_m, or

b) at least 2log(m)+1 points of PConv(P_n) which constitutes one of the 2^i subsets of the vertices of PConv(P_n)

Since the splitting process is repeated i times, Φ_q contains at most i=log(m) edges which arise from the lines used to split Q_m (See figure 4 (a).) It is easy now to verify that a) and b) together imply that in each Φ_q there are three consecutive vertices of PConv(P_n) (See Figure 4(a).)



(a)



(b)

Figure 4.

Now in each Φ_q we can find a poygon enclosing q as in Lemma 2 (See Figure 4(b).) Finally joining all of the previously obtained polygons and deleting the lines used to split Q_m we obtain our desired P_n-polygon enclosing Q_m. This ends our proof.

References

[1] H. Edelsbrunner, "Algorithms in Combinatorial Geometry" Springer Verlag (1987)