Onion Polygonizations *

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Abstract

In this paper we introduce the class of *onion polygons*, specially intended for solving the following problem: how to polygonize a set of points in such a way that when the convex hull is removed, the remaining points can be polygonized quickly with minimum changes. In that sense onion polygons can handle even more complicated situations. Moreover, they also enjoy special good properties in other computational aspects.

1 Introduction

A *polygonization* of a set of points is a way of connect them to form a simple polygon. This problem has been a recurring theme in Pattern Recognition and Computational Geometry [3,8,10], its complexity being $\Omega(n \log n)$. Two main objectives are to capture the shape of the points and to obtain polygons with nice properties from the computational point of view. Newborn and Moser [7] proved that the number of polygonizations of a set of n points can be exponential in n, and their bounds were later improved [1,4]. Meijer and Rappaport [6] proved that this number remains exponential when restricted to monotone polygons, as did Deneen and Shutte [3] for the (degenerate) star-shaped case. Monotone and star-shaped polygonizations can be constructed in time $O(n \log n)$.

The convex layers of a set of points are the convex polygons obtained by successively "peeling" the convex hull of the set [2,9] . This process has become central in many problems in statistics after a suggestion by Tukey since estimators can be extremely sensitive to outliers [5]. It also provides valuable information on the morphology of a set of sites and has proven to be an efficient preconditioning for range search problems. Chazelle [2] proved that convex layers can be computed in optimal time $O(n \log n)$.

When a set of points is polygonized, the "peeling" process destroys the polygon possibly in a dramatic way. This suggests the following problem: how to polygonize a set of points in such a way that when the convex hull is removed, the remaining points can be polygonized quickly with minimum changes. In this paper we introduce the class of *onion polygons* to solve this problem. More precisely, after the removal of any number of consecutive layers, the remaining set can be polygonized in constant time; moreover, not destroyed adjacencies are preserved and the resulting polygon is again an onion polygon, therefore allowing subsequent iterations. Onion polygons are constructed in $O(n)$ time if the convex layers are available, so resulting in a total $O(n \log n)$ time if we are given the naked set of points. One way to relate the convex layers and the polygonizations of a set of points is by means of the *depth* of a point of the set. The depth of a point p in a set S is the number of convex layers that have to be stripped away before p is removed. Every polygonization of the set produces a sequence of depths. We show that onion polygonizations correspond to unimodal sequences. In a similar way, we obtain the notion of the *weigth* of a polygon when every edge is weighted by the absolute difference of the depths of its extremes. We show that onion polygonizations have minimum weigth. Some problems related to

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these concepts are also raised. In a different direction, the strongly recursive structure of onion polygons make them very suitable in many computational aspects. The paper is organized as follows. Next Section deals with depth sequences and weigth of polygons. In Section 3, onion polygons are introduced and its properties are studied. We conclude with some remarks and open problems.

2 Depth sequence and weigth of a polygon

Let S be a set of n points in the Euclidean plane. The convex layers of S are the convex polygons obtained by iterating on the following procedure: compute the convex hull of S and remove its vertices from S. The ***depth*** of a point p in S is the number of convex layers that have to be stripped away before p is removed. The depth of S is the depth of its deepest point. If P is a polygon, we define the ***depth sequence*** of P as the (circular) sequence $\mathcal{A}$ of the depths of its vertices. We also say that $\mathcal{A}$ is ***fulfilled*** on P. The elements of the depth sequence of a polygon are integers from 1 to k, where k is the number of convex layers, every integer from 1 to $k-1$ appearing at least three times. From now on all sequences we consider satisfy this property for some k, so that they could represent the depth sequence of some polygon, and have first term 1. We call the ***jump*** of the sequence the maximum of the absolute value of the differences between consecutive terms. Natural questions are

a) Given such a sequence $\mathcal{A}$, is there any polygon fulfilling $\mathcal{A}$?

b) Given a set of points S and a permutation $\mathcal{A}$ of the depths of its points, is $\mathcal{A}$ fulfilled on S, that is, do there exists any polygonization of S fulfilling $\mathcal{A}$?

Partial answers to the precedent questions can be obtained by considering the special class of ***unimodal*** sequences. A sequence of numbers $\{a_1, \ldots a_n\}$ is called unimodal if for some integer t, $a_1 \leq \cdots \leq a_t$ and $a_t \geq \cdots \geq a_n$. For these sequences the answer to question a) is affirmative.

Proposition 2.1 *Let $\mathcal{A}$ be a unimodal sequence. There exists a polygon P which fulfills $\mathcal{A}$.*

For a fixed set of points S not every permutation of its depths is always fulfilled on S, even with the restriction of unimodality. A counterexample is shown in Figure 1.

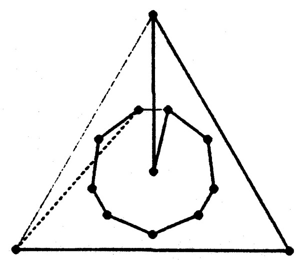

Figure 1: The sequence 1322222222211 can not be fulfilled.

This example can also be used to construct sets and unimodal sequences of jump greater than 1 the set can not fulfill. The situation is quite different when the jump is 1.

Proposition 2.2 *Let S be a set of n points and $\mathcal{A}$ a permutation of the depths of its points. If $\mathcal{A}$ is unimodal with jump 1, then it is fulfilled on S. Moreover, if the convex layers of S are given, the construction of the polygonization can be c arried out in $O(n)$ time.*

For future convenience, the polygonization constructed in the proof possess an additional property: edges connecting consecutive layers arrive to and quit from vertices that are consecutive in each layer.

The depth sequence of a polygon gives rise to another interesting concept. We define the ***weigth*** of an edge of a polygon as the absolute value of the difference of the depths of its extreme points. The ***weigth*** of a polygon is the sum of the weigths of its edges. A convex polygon has weigth 0. From an intuitive point of view, the greater the weigth is, the further we are from the convex ideal, so weigth could be used as an additional measure of complexity shape.

Let S be a set of n points and depth k. As $k \leq \frac{n}{3}$ and every edge can have weigth at most $k-1$, a rough upper bound for the weigth of a polygonization of S is $n(k-1) \leq \frac{n^2}{3} - n$. Figure 2 shows a polygon of n points and weigth $\frac{n^2}{6}$.

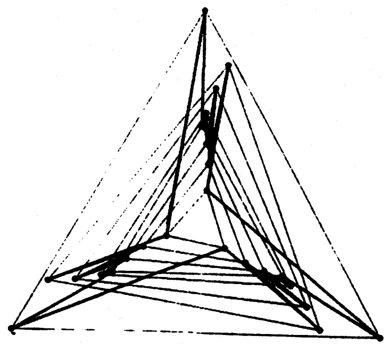

Figure 2: A polygon with depth $2q$, $n = 6q$ vertices and weigth $\frac{n^2}{6}$.

On the other hand, we have the lower bound $2k-2$ since each convex layer must be visited, this bound being attained by the polygonizations constructed in Proposition 2.2.

Proposition 2.3 *Let S be a set of n points and depth k. A polygonization of S has the minimum weigth $2k-2$ if and only if its depth sequence is unimodal.*

3 Onion Polygons

In the proof of Proposition 2.2 we have constructed minimum weigth polygons of jump one by connecting two consecutive vertices of each layer to two consecutive vertices of the following one. It is not a necessary condition for the vertices of the internal layer to be consecutive but we have done this construction because of the good properties of the family of polygons obtained in this way.

A simple polygon is an *onion polygon* if its depth sequence is unimodal, it has jump equal to one and the two edges that connect every two consecutive convex layers of the set of vertices, connect two consecutive vertices of a layer to two consecutive vertices of the following one (see Figure 3).

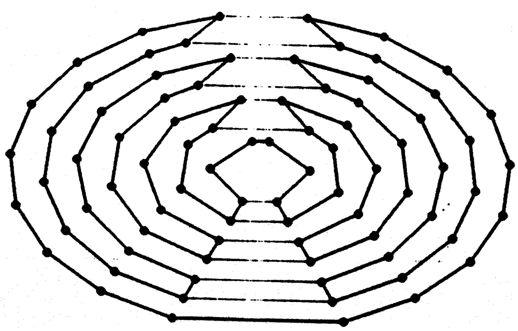

Figure 3: An onion polygon.

Observations.

1. In an onion polygon there are two and only two edges connecting two consecutive convex layers of its vertices because of the conditions of unimodality and jump imposed to the depth sequence .
2. From proposition 2.2 it follows that it is possible to obtain an onion polygonization of a set of n points in linear time with $O(n \log n)$ preprocessing time to obtain the onion of the set.
3. Onion polygons have minimum weigth because of proposition 2.3.

Onion polygons form an interesting family of polygons as well as star-shaped or monotone polygons. In [3] nondegenerate star-shaped polygonizations are studied by Deneen and Shute and it is shown that $O(n^4)$ such polygonizations exists for a set of n points in general position. Meijer and Rappaport in [6] have obtained the following bounds for the number $\Psi(n)$ of monotone polygonizations of a set of n points: $(2+\sqrt{5})^{\frac{(n-3)}{2}} \leq \Psi(n) \leq (\sqrt{5})^{(n-2)}$. For the onion polygonizations we have proved the following:

Proposition 3.1 *Let $C(n)$ be the maximum number of onion polygonizations that can be constructed for a set of n points. Then,*

$$\frac{3}{20}10^{\frac{n}{3}} \leq C(n) \leq 10^{\frac{n}{3}}$$

In spite of the exponential number of onion polygonizations, it is possible to obtain an enumerative algorithm with linear storage requirement whose output is the list of all onion polygonizations of the set of points given in the input as we prove :

Proposition 3.2 *The whole list without duplications of onion polygonizations of a set of n points can be generated using $O(n)$ storage.*

Proposition 3.3 *Whether a simple polygon is or is not an onion polygon can be recognized in linear time . Moreover, if the polygon is an onion polygon, then the depths of its vertices can also be obtained in linear time.*

The most interesting properties of onion polygons are due to its recursive structure relative to the convex layers of the set of vertices. An onion polygon can be immediately decomposed in two onion polygons, the second being completely contained inside the deepest convex layer of the first. This recursive structure allow us to obtain simple specific algorithms to solve problems on these polygons such as "watchman route", "scape of the maze" or the construction of the d-neighbourhood of the polygon. To triangulate these polygons there also exists a very simple linear algorithm making use of Toussaint's algorithm [11] to triangulate the onion of a set of points.

Another interesting property of the onion polygons is that its convex deficiency is an onion polygon if we remove the quadrilateral lying between the first two convex layers of the vertices of the polygon.

The main property of onion polygonizations is related with the process of *onion peeling*. This classic statistical problem consists in the removing of the outliers of a set of points; this is usually done by removing several convex layers of the onion. If the point set has been polygonizated before cleaning the outliers, then to delete the extreme vertices of the set gives rise to the disconnection of the polygon in such a way that it is expensive to reconnect it again. The excellent behaviour of onion polygonizations face to that problem is stated in the following theorem.

Theorem 3.4 *Let $\mathcal{P}$ be an onion polygonization of a point set of depth k. If the vertices of depth lower than i $(0 \leq i \leq k)$ are removed from the polygon, then an onion polygonization of the rest of the set can be obtained in constant time.*

Observations.

1. The constant time reconstruction algorithm needs a preprocessing algorithm which runs in linear time to locate the gates of each layer of the polygon. This preprocess can be included in the initial polygonization process of the set maintaining its linear behaviour.
2. Moreover, with another additional $O(n)$ preprocessing time, that can be too included in the initial process, it is possible to obtain a constant time algorithm to delete every set of consecutive layers.
3. If a consecutive set of vertices of a layer, but not the whole layer, is removed, then it is possible to reconstruct the polygonization in linear time (take into account that such a deletion can give rise to the modification of all the convex layers of the set).

4 Concluding Remarks

As we show in the paper, onion polygons allow us to solve questions such as to fulfill an unimodal depth sequence with jump 1 on a fixed set of points, to compute minimum weigth polygonizations and to obtain in linear time the depths of the vertices of a polygon once recognized to be an onion polygon (such a recognition achieved within the same bound).

Open related questions are,

1. What families of depth sequences on a set S can be fulfilled?
2. How to polygonize a given set to obtain maximum weigth?
3. If we are given a polygon P, can the depths of the vertices and the weigth of P be obtained in time under $n \log n$?

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