Complexity of Projected Images of Convex Subdivisions

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1 Introduction

Projection and the projected images often play important roles in algorithms for computational geometry. Let S be a subdivision of $\mathcal{R}^d$ into n convex regions. In this paper, we investigate the complexity of the image $Pr_k(S)$ of the $(d-k-1)$-skeleton of S orthogonally projected into a $(d-k)$-dimensional subspace.

A typical application is the design of data structures for point location in a subdivision of a space [Co, DL, PT, THI]. The efficiency usually depends on the complexity of the images of the subdivision projected into lower-dimensional spaces. In the three-dimensional case, two data structures for locating a point in a convex subdivision S are known.

One has a query time of $O(\log n)$, and needs the same complexity for the space as for the image $Pr_1(S)$ of S projected onto a plane [THI]. The other has a query time of $O(\log^2 n)$, but reduces the space complexity to $O(Pr_2(S)\log^2 n)$ [PT]. Naturally, the complexity of $Pr_2(S)$ is the same as the number K of vertices in S. In order to compare these two methods, we must estimate the complexity of $Pr_1(S)$. If we estimate the complexity of $Pr_1(S)$ with respect to K, it can be $\theta(K^2)$. Since K is $\theta(n^2)$, a naive upper bound of $Pr_1(S)$ is $O(n^4)$ faces. On the other hand, if we estimate $Pr(S)$ with respect to n, it is not trivial to construct a subdivision S such that $Pr_1(S)$ contains more than $\Omega(n^2)$ faces.

In this paper, first we show that the complexity of the projected image $Pr_1(S)$ is $\theta(n^3)$ for a three-dimensional subdivision S, and generalize the result to higher-dimensional cases.

In the three-dimensional case, the problem is also related to the visualization of a convex subdivision of three-dimensional space. This problem is interesting, since a 3D-Voronoi diagram, which is a typical example of convex subdivision, is a popular tool for simulating objects in nature. One natural method of realizing the above visualization is to animate of $Pr_1(S)$ by rotating the projection plane. We show a $\theta(n^4)$ bound for the number of topological changes of the projectied image if a three-dimensional subdivision is rotated about a line in the projection plane.

2 Three-Dimensional Case

Let S be a convex subdivision of three-dimensional space into n polytopes. The number of edges in S is denoted by K. $Pr_1(S)$ ($Pr(S)$ for short) is the projected image of the 1-skeleton of S onto a plane H.

Theorem 2.1 *The number of vertices in $Pr(S)$ is $O(nK)$.*

Proof The regions are numbered as Q_i for $i = 1, 2, .., n$. For each region Q_i in S, the boundary of the convex hull of the projected image $Pr(Q_i)$ is denoted by $C(Q_i)$, and number of edges of

$C(Q_i)$ is denoted by k_i. Every edge of $C(Q_i)$ is a projection of and edge of Q_i, and each edge contributes to at most two $C(Q_i)$, so $\sum_{i=1}^{n} k_i \leq 2K$. For each edge e in S, let $H(e)$ be the plane containing e and perpendicular to H. Because of the convexity, there exists a region Q containing e that does not intersect $H(e)$. Apparently, $Pr(e)$ is an edge of $C(Q)$. Therefore, every projected image of an edge is contained in the convex boundary of a projected image of a suitable region. Since $C(Q_i)$ and $C(Q_j)$ intersect at at most $2\min(k_i, k_j)$ points, there are at most $2\sum_{i,j=1}^{n} \min(k_i, k_j) \leq 2n\sum_{i=1}^{n} k_i \leq 4nK$ intersections in $Pr(S)$. □

Corollary 2.2 *The number of vertices in $Pr(S)$ is $O(n^3)$.*

Next, we give the lower bound of the complexity of the projection of the skeleton of a three-dimensional convex subdivision satisfying the condition that K, the number of edges of the convex subdivision S, is at least n. We fix a plane H in the space.

Theorem 2.3 *There exist constants c and c' such that for any $cn^2 \geq K \geq c'n$, there exists a convex subdivision S satisfying the following conditions: (1) S has n convex polytopes, (2) S has K vertices in total, and (3) the projection $Pr(S)$ of S on H has $\Theta(nK)$ vertices.*

Proof It suffices to show the lower bound. From the Dehn-Sommerville equation, $K \leq n^2$. For a given arbitrary number $18n \leq k \leq n^2$, we set $m = k/6n$, and $s = (n - 2m)/2$. It is easy to see that $s > n/3$. We consider a circle C on a plane H_1 parallel to H. Let l be a line through the center O_1 of C perpendicular to H. Let A_1 denote the set of vertices of a regular m-gon inscribed into the circle C. Further, we consider a point set B_1 consisting of s points on l. We also consider the Voronoi diagram V_1 of $A_1 \cup B_1$. The projection of the cap boundary of each region of a point of B_1 in V_1 is a regular m-gon with center O, which is the projection of O_1. Further, each of these polygons can be transformed into another by a scaling transformation. If the whole set of B_1 is close enough to O_1, the scaling factor is larger than $\cos\frac{\pi}{2m}$ (and smaller than 1) for each pair.

Next, we consider another plane H_2 parallel to H, and project A_1 onto H_2 to obtain a point set on the circle C_2 with center O_2. We rotate these points on C_2 by an angle $\frac{\pi}{m}$ to obtain a point set A_2. We let a point set B_2 be a translation of B_1 by the vector $\vec{O_1O_2}$. The Voronoi diagram V_2 of $A_2 \cup B_2$ is congruent to V_1. However, since the point set is rotated, the projection of the cap boundary of the region of a point of B_2 is rotated by $\frac{\pi}{m}$ with respect to the corresponding one in V_1.

Now, we place H_2 sufficiently far from H_1, and consider the Voronoi diagram V of $A_1 \cup A_2 \cup B_1 \cup B_2$. The projection of the cap boundary of the region of a point of B_1 intersects at $2m$ points with that of any point of B_2. Since there are s^2 pairs of such cap boundaries, the total number of intersections is at least $2ms^2 \geq \frac{1}{27}nk$.

On the other hand, the Voronoi diagram V has n regions and $K = \frac{k}{9} + O(n)$ edges. Thus, we obtain the theorem. □

In the above theorem, we fix the projection plane H. However, more generally, the following holds:

Theorem 2.4 *There exists a convex subdivision S of the space into n convex regions such that its projected image onte any plane has $\Omega(n^3)$ vertices.*

The proof is omitted in this version.

3 Rotation and topological change

In this section, we investigate the topological change of $Pr(S)$ when S is rotated about a line.

If the projected image is used for the point location structure, the complexity of the projected image, which coinsides the space complexity of the point location structure, should be reduced as much as possible. If we can cheaply rotate the subdivision, we can find the angle such that the complexity of the projected image is minimum. Moreover, if we consider the visualization of an object, we often need to rotate the object and make an animation.

A topological change occurs when three projected edges meets. Since there are $K = O(n^2)$ edges, a naive bound of the number of topological changes is $O(n^6)$.

We say that a rotation is parallel if the rotation axis is parallel to the projection plane. Otherwise, it is called a skew rotation.

We can prove the following theorem:

Theorem 3.1 *The number of topological changes is* $\Theta(n^4)$ *for a parallel rotation.*

Proof First, we prove the upper bound. We assume that the projection plane H is parallel to the $x-y$ plane. The plane H_Θ is obtained by rotating H by about the x-axis by an angle θ. The projection Pr_θ is orthogonal to H_θ. Suppose the projected images of three edges e_1, e_2, and e_3 meet at a point $p_0 = (x_0, y_0, z_0)$ of H_Θ. Then, we consider the plane $H(x_0)$ intersecting the x-axis orthogonally at x_0. The intersecting points of e_1, e_2, e_3 are located on a line on $H(x_0)$.

Let $X(S) = \{x_1, x_2, .., x_N\}$ be the sorted list of the x-coordinate values of the vertices of S. We define $x_0 = -\infty$ and $x_{N+1} = \infty$. Let us count the topological changes by using a space sweep with respect to the x-axis. We consider a sweep plane $H(t)$, which intersects the x-axis orthogonally at $(t, 0, 0)$. Let $S(t)$ be the intersection of S with $H(t)$. Obviously, $S(t)$ is a planar convex subdivision with $O(n)$ regions; thus it has $O(n)$ vertices. We move the sweep plane $H(t)$ from $t = x_0$ to $t = x_{N+1}$, and count the number of colinear triples of the vertices of $S(t)$. Let x_i and x_{i+1} be two consecutive elements of $X(S)$. For any two values t and t' in (x_i, x_{i+1}), the graph structure of $S(t')$ is the same as that of $S(t)$. For each triple e_1, e_2, e_3 of edges, there is at most one t' such that the points of intersection with $H(t')$ are located on a line. Thus, at most n^3 topological changes are found during the sweep from $x_0 = -\infty$ to x_1. When the sweep plane passes through $x = x_i$, k_i edges (incident to the vertex corresponding to x_i) are newly cut by the sweep plane. Thus, $k_i n^2$ triples are newly created. Since $\sum_{i=1}^{N+1} k_i = O(n^2)$, the total number of topological changes is $O(n^4)$.

Next, we consider the lower bound. We use the Voronoi diagram V defined in Section 2 and adopt the notations used there. We assume that the distance D between H and H_1 is sufficiently large, and that the distance between H_1 and H_2 is very small. Let us assume that the rotation axis α on H meets the line l at the origin O. Let $\bar{l}$ be a line on H orthogonal to α, such that $\bar{l}$ meets l and α at O. We define a set X of n points on $\bar{l}$, such that the distance between two extremal points of X is δ. Let $\bar{V}$ be the Voronoi diagram of the point set $A_1 \cup A_2 \cup B_1 \cup B_2 \cup X$. We consider the subdivision S that arises when the plane H is added to $\bar{V}$. Evidently, S is a convex subdivision consisting of $O(n)$ regions. Since H_1 is far enough from H, almost all regions of V survive in S (actually, only the lower envelope of V is changed). We call this part $\tilde{V}$. There exists a maximal angle ϕ such that the topological structure of $Pr(\tilde{V})$ is not changed if we rotate it by any angle between $-\phi$ and ϕ. This angle ϕ is independent of the distance D between H and H_1. On the other hand, S contains the set $\mathcal{L}$ of $n-1$ segments parallel to α, which are intersections of the plane H and the Voronoi boundary of the points of X. The maximal distance between them is bounded by δ. We can assume that $D \tan\phi > \delta$. Then, during the rotation of S from $-\phi$ to ϕ, each of the $\Theta(n^3)$ vertices of $Pr(\tilde{V})$ meets each of the $n-1$ segments of $\mathcal{L}$ at an angle. Thus, there are $\Omega(n^4)$ topological changes. □

Any rotation is written as a product of the three rotations about the x-axis, y-axis, and z axis. Obviously, the rotation about z-axis causes no topological change in $Pr(S)$. Thus, a skew rotation for a given angle is represented as a product of parallel rotations. However, the bound in Theorem 3.1 may fail for the number of topological changes occurring during a skew rotation about a fixed axis if we consider the angle as a continuous parameter. In dealing with a skew rotation, we should consider the intersection of the subdivision with a circular cone instead of with a hyperplane. Unfortunately, the complexity of an intersection of S with a circular cone is $\Theta(n^2)$. Therefore, it remains an open problem to obtain a nontrivial bound for the number of topological changes for a skew rotation.

It may be remarked that if we count the number of possible different topologies with respect to all three-dimensional rotations (that is, the orthogonal group of the space), the bound is known to be $O(K^6)$ [S]. If we naively substitute $K = O(n^2)$ into this, we get a complexity $O(n^{12})$.

4 Higher-dimensional extension

In this section, we investigate convex subdivisions in higher-dimensional spaces and show an upper bound and a lower bound for the complexity of the projected image.

Let S be a convex subdivision of R^d into n polytopes. It is well-known [E] that the worst-case complexity of S is $\Theta(n^{\lfloor (d+1)/2 \rfloor})$. The projection of the $(d-k+1)$-skeleton of S onto a $(d-k)$-dimensional subspace L is denoted by $Pr_k(S)$.

A face of S is called *facet* if it has codimension 1. A face of dimension j is called a j-face.

Any face of $Pr_k(S)$ is an intersection of projected images of at most $d-k$ faces of S. The projection is called nondegenerate if there are no degenerations in $Pr_k(S)$ except those originally in S. It is easy to observe the following lemma:

Lemma 4.1 *The complexity of $Pr_k(S)$ is asymptotically bounded by the number of vertices in $Pr_k(S)$ and the number of original faces if the projection is nondegenerate.*

To obtain the upper bound of the complexity of $Pr_k(S)$, we can assume without loss of generality that the projection is nondegenerate.

Let f_i be the number of faces of dimension i of S.

Theorem 4.2 *The complexity of the image $Pr_k(S)$ of the $(d-k-1)$-skeleton of S projected into an affine subspace of codimension k is $O((f_{d-k-1})^{\lceil (d-k)/2 \rceil}(f_{d-k+1})^{\lfloor (d-k)/2 \rfloor})$*

Proof Let P be a $(d-k+1)$-face of S. The boundary $B(P)$ of $Pr_k(P)$ is a convex polytope in R^{d-k}. Let f be an arbitrary $(d-k-1)$-face of S. There exists a hyperplane H, which is perpendicular to the projection subspace L and contains f. It is easy to see that there exists at least one $(d-k+1)$-face P of S located in one of the half-spaces defined by H. Obviously, $Pr(f)$ is contained in $B(P)$. In fact, $Pr(f)$ is a facet of $B(P)$. Let $f(P)$ be the number of facets of $B(P)$. If a facet of $B(P)$ is shared by another polygon $B(Q)$, we erase the facet from $B(P)$, and reconstruct the polygon from the hyperplanes associated with the remaining facets. Thus we can assume that the summation of $f(P)$ over all of P is $O(f_{d-k-1})$.

Thus, $Pr_k(S)$ is an arrangement (in R^{d-k}) of f_{d-k+1} convex polytopes, and the sum of the number of their facets is $O(f_{d-k-1})$. It is known that the complexity of an arrangement of N convex polytopes with M facets in D-dimensional space is $O(N^{\lceil D/2 \rceil} M^{\lfloor D/2 \rfloor})$ [ABE]. Thus, we obtain the theorem. □

As a special case, let us consider $Pr_1(S)$.

Corollary 4.3 *The complexity of $Pr_1(S)$ is $O(n^{2d-3})$ if d is even, and $O(n^{2d-2})$ if d is odd. Moreover, the complexity is $O(n^d)$ if $d \leq 4$.*

This time complexity is better than the naive $O(n^{3(d-1)})$ bound by a factor of n^d or n^{d-1}. A naive lower bound of the complexity of $Pr_1(S)$ is $O(n^{d-1})$. We give a better lower bound below:

Theorem 4.4 *The complexity of $Pr_1(S)$ is $\Omega(n^{\lfloor(3d-3)/2\rfloor})$.*

Proof Let us consider the moment curve $\Gamma : x(t) = (t, t^2, t^3, .., t^{d-1})$ of R^{d-1}. We consider a set $M = \{x(\tau_i) : i = 1, 2, ..n\}$ of $(d-1)n$ points on Γ. We assume that $\tau_i < \tau_j$ if $i < j$. The convex hull of M is denoted by $C(M)$. It is well known that $C(M)$ has $\Omega(n^{\lfloor(d-1)/2\rfloor})$ facets. The subset M_i of M is defined by the set $\{x(\tau_j) : j \equiv i \pmod{d-1}\}$. We cluster M into $d-1$ subsets $M_1, M_2, .., M_{d-1}$.

Let us investigate the facets in detail. An index set $I = \{i_1, i_2, ..., i_{d-1}\}$ of size $d-1$ is called *special* if $I \subset \{1, 2, .., n\}$ and $i_j = i_{j-1} + 1$ if j is even. Furthermore, we set $i_{d-1} = n$ if $d-1$ is odd. We define the function $f_I(x) = \prod_{j \subset I}(x(\tau_j) - x)$. It is easy to see that this function is nonnegative on M, and zero on $x(\tau_j)$ if $j \in I$. Since the degree of $f_I(x)$ is $d-1$, from a similar argument to the one on p.101 of [E], $f_I(x) = (u, x) - v_0$ on Γ for suitable vectors u and v_0. Hence, the hyperplane spanned by $\{x(\tau_j) : j \in I\}$ appears as a facet of $C(M)$.

Thus, there are $\Omega(n^{\lfloor(d-1)/2\rfloor})$ facets of $C(M)$, each of which is spanned by a point set containing exactly one point of each subset M_i $(i = 1, 2, .., d-1)$.

Let $D(M)$ be the set of dual hyperplanes of M, and let $D(C(M))$ be the dual of $C(M)$. We choose a point x in the interior of $D(C(M))$. For each hyperplane h in $D(M)$, the point opposite to x with respect to h is denoted by $x(h)$. The point set $\{x(h) : h \in D(M_i)\}$ is denoted by $\hat{M}_i$.

Let g be the d-th axis of R^d. We choose the points x_i $i = 1, 2, .., d-1$ such that the distance between each pair of these points is sufficiently large. We consider the hyperplane L_i orthogonal to g containing x_i. Now, we translate the point set $\hat{M}_i$ so that x is translated to x_i. We generate n points on g that are infinitesimally near to x_i. Let us denote V_i for the Voronoi diagram generated by these $2n$ points. V is the merged Voronoi diagram of V_i for $i = 1, 2, .., d-1$. The Voronoi region of a point on g is called a central region. Then, if we select a central region from each cluster, the intersection of these $d-1$ regions contributes $\Omega(n^{\lfloor(d-1)/2\rfloor})$ vertices because of the claim. Thus, we obtain the theorem. □

From Theorem 4.4, the upper bound of Corollary 4.3 is tight if $d \leq 4$.

5 Algorithmic aspect

The proof of Theorem 4.2 gives an algorithm for computing $Pr_1(S)$, which runs in $O(n^{2\lfloor(d-1)/2\rfloor+d-1})$ time when the optimal convex hull algorithm of [C] or [S1] is used. With more precise analysis, this algorithm runs in $O(n^3)$ or $O(n^4 \log n)$ time if dimension is three or four, respectively.

The output size is usually much smaller than the worst-case size; thus, an efficient output sensitive algorithm is desirable. The plane sweep method solves the problem in $O(M \log n)$ time if $d = 3$ (where M is the output complexity). Further, if we use the optimal segment intersection reporting algorithm [CE], an $O(M + K \log n)$ time algorithm can be designed, where K is the number of edges in S.

In the four-dimensional case, the projected image is an arrangement of n convex polyhedra in three-dimensional space. The total number of faces of the polyhedra in the arrangement is $O(n^2)$. Below, we give an $O(M \log n)$ method.

Let us consider the space sweep method for computing $Pr_1(S)$. We consider the sweep plane $x = t$ orthogonal to the x-axis, and translate it from $t = -\infty$ to $t = \infty$. The intersection $\Sigma(t)$ of $Pr_1(S)$ with the plane $x = t$ is an arrangement of convex polygons. The complexity of $\Sigma(t)$ is $O(Min(n^3, M))$. For each edge e of $\Sigma(t)$, we compute the value of t at which the edge vanishes. For all such edges, we keep these values in a priority queue. We update this priority queue during the sweep. If the sweep comes to the abscissa of a vertex of S, more than one element of the priority queue may be updated. However, the total number of priority queue operations is $O(M)$ during the sweep. Therefore, the sweep method gives an $O(M \log n)$ time algorithm for computing $Pr(S)$.

For higher-dimensional cases, an output-sensitive algorithm for computing a convex hull in $O(n^2 + h \log n)$ time has been developed by Seidel [S2], where h is the number of faces on the convex hull. Let k_i be the number of vertices of $Pr(S)$ that lie on the projected images of a $(d-2-i)$-dimensional face of S. If we apply Seidel's output-sensitive convex hull algorithm, we obtain a slightly output-sensitive algorithm. The time complexity is at most $O(n^{d+1}) + \sum_{i=0}^{d-2} n^i k_i \log n$. It is easy to see that $k_i = O(n^{2\lfloor((d-1)/2)\rfloor + d - 1 - i})$ and much smaller in practice.

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