ON CLASSES OF ARRANGEMENTS OF LINES

(EXTENDED ABSTRACT)

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ABSTRACT

We introduce a hierarchy of classes of arrangements of lines based on the number of convex vertices of their envelopes. We show how the properties of envelopes enable us to determine in O(n) time if a polygon of n vertices is the envelope of an arrangement of lines of a given class. In particular, we look at a class called *sail arrangements* which we prove has properties that enable us to solve a number of problems optimally. Given a sail arrangement consisting of n lines (and $O(n^2)$ vertices), we show how the Megiddo technique can be used to determine all the convex vertices of its envelope in O(n) time. This implies that the intersection point with minimum or maximum x-coordinate, the diameter and the convex hull of sail arrangements can also be found in O(n) time. We show, however, that the problem of constructing the envelope of a sail arrangement has a lower bound of $\Omega(n \log n)$. We also examine the existence of hamiltonian circuits through the intersection points of a non-trivial subclass of sail arrangement graphs.

1. Introduction

In [CL85], Ching and Lee established an $\Omega(n \log n)$ lower bound for the problem of computing the envelope (essentially the outer boundary) of an arrangement of lines under the algebraic tree computation model. Suri [Su85], Vegter [Ve87] and Keil [Ke91] contributed algorithms that exhibit $O(n \log n)$ time complexity for this problem (note that Keil's algorithm runs in O(n) time given the lines of the arrangement sorted in order of slope). While optimal running time algorithms have been found for computing the envelope of an arrangement of lines, little has been done to solve the recognition problem. That is, given a simple polygon P of n edges, how quickly can we determine whether or not P is the envelope of an arrangement of lines? Using a brute force approach, we can answer this question in $O(n \log n)$ time. We first extend all of the edges of P to obtain an arrangement A. The envelope of A is then computed using any one of the algorithms in [Su85], [Ve87] or [Ke91] and then compared to P (points must correspond). The simplicity of this algorithm seems to suggest that an algorithm that exploits some characteristic property of envelopes could improve on the upper bound of $O(n \log n)$.

In this paper we introduce a hierarchy of classes of arrangements of lines based on the number of convex vertices of their envelopes. This approach has proven to be a productive alternative in the field of computational geometry as witnessed in [ET89], where much progress was made as a result of defining a hierarchy of polygons that possess more structure than arbitrary simple polygons. We show how the properties of envelopes allow the recognition problem for a given class to be solved in O(n) time, where n is the number of lines in the input polygon, by using Keil's algorithm. We also show that a certain class of arrangements, which we call *sail arrangements*, has properties that allow us to determine the convex vertices of sail envelopes in O(n) time given a sail arrangement of n lines. Consequently, we can solve several other problems regarding the $O(n^2)$ intersection points of a sail arrangement in O(n) time. In particular we can find the intersection point with minimum/maximum x-coordinate as well as the diameter and convex hull of the intersection points. These problems where shown to have $\Omega(n \log n)$ lower bounds for arbitrary arrangements under the algebraic tree computation model in [CSSS89] and [CL85] respectively. We show, however, that even if we know the convex vertices of the envelope of a sail arrangement, computing the remainder of the envelope is Ω(n log n) under the algebraic computation tree model. We also show that there exist non-trivial sub-classes of sail arrangements which always admit a hamiltonian cycle, and furthermore, we exhibit a polynomial time algorithm to compute one. To satisfy page-number limitations we omit all proofs in this extended abstract.

2. Definitions

Let L_i and L_j be two non-parallel lines. Denote their intersection point as $I(L_i,L_j)$. Let $A = \{L_0,L_1,\dots,L_{n-1}\}$ be an arrangement of n lines. We shall denote the set of intersection points of the lines of A by *S(A)*. Formally, $S(A) = \{I(L_i,L_j) \mid L_i,L_j \in A, i \neq j\}$. Point $p = I(L_i,L_j)$ ($i, j \in [0,n-1]$) is said to be ***extreme*** on L_i if every intersection point lying on L_i lies on one side of p. The point p is said to be ***critical*** if it is extreme on both L_i and L_j. Two non-parallel lines in an arrangement are said to be ***adjacent*** if they are neighbors in the list of the lines of the arrangement sorted by slope. The ***envelope*** of an arrangement A, denoted as E(A), is the simple polygon formed by the bounded line segments of the unbounded planar subregions (or faces of A) induced by the arrangement of lines.The ***induced arrangement*** of a polygon P, denoted as IA(P), is the arrangement obtained by extending the edges of P to lines. A simple polygon P is an ***envelope*** if P = E(IA(P)).

3. Geometric Properties of Envelopes

In this section, we establish that envelopes are L-convex. We strengthen this result by first showing that non-trivial envelopes (envelopes of arrangements of more than three lines) are not convex. We also characterize the convex vertices of envelopes with the following lemma that will be useful in subsequent sections.

Lemma 3.1. Let A be an arrangement of lines. A vertex of E(A) is convex if and only if it is a critical vertex of S(A).
Corollary 3.1.1. If P is an envelope then the convex vertices of P are the intersection points of adjacent lines in IA(P).

We are interested in determining the smallest well-know class of polygons that subsumes the set of polygons that are envelopes of arrangements. We start by showing that non-trivial envelopes (envelopes of arrangements of more than three lines) are not convex.

Lemma 3.2. If P is an envelope, then P cannot have more than three consecutive convex vertices.
Corollary 3.2.1. An envelope is convex if and only if it is a triangle.

In [Za75], Zaslavsky conjectured that the envelope of an arrangement of lines was star-shaped; a class of polygons that subsumes the class of convex polygons. Vegter [Ve87] showed that this is not the case. We can, however, show that envelopes are L-convex. We define L-convexity as an instance of L_k-convexity as follows. A simple polygon P is ***L_k-convex*** ($k \geq 1$) if for every two ***points*** x and y that lie inside P, there exists a set of k-1 distinct points $L = \{q_1,q_2,\dots,q_{k-1}\}$ in P such that the k line segments $[x,q_1], [q_1,q_2],\dots,[q_{k-2},q_{k-1}],[q_{k-1},y]$ all lie inside P. Note that for $k = 1$, the set L is empty and we obtain the classical definition of a convex polygon.We can think of L as a polygonal chain of k line segments that lies in P and has x and y as endpoints. If a set L exists, we say that x and y have a ***link distance*** of k with path L. We say that x and y have ***minimal link distance*** k if k is minimal over all possible link paths between x and y. We say that a polygon P is ***L-convex*** if it is L_2-convex.

Lemma 3.3. A simple polygon P is L_k-convex if and only if the minimal link distance between every two ***vertices*** of P is at most k.
Lemma 3.4. Let A be an arrangement of n lines, then E(A) is L-convex.

Whereas envelopes of arrangements have not been studied much, L-convex sets have received considerable attention [HV49]. Properties of L-convex polygons have been exploited in [EAT83] to obtain efficient algorithms for solving a variety of geometric problems. Since envelopes are L-convex, properties of L-convex polygons are useful for answering questions about envelopes. In the following section we establish a new property of L-convex polygons.

4. A Property of L-convex Polygons

We present a property of L-convex polygons that is useful in characterizing classes of arrangements. Suppose that $C = [p_u, p_{u+1}, ..., p_v]$ is a clockwise vertex chain of some simple polygon P. Let the interior of the polygon be the region to the right of C as we move from p_u to p_v on C. Let R_u be the directed line colinear to $[p_{u+1}, p_u]$. Let R_{i+1} be the directed line obtained by rotating R_i counterclockwise about p_{i+1} until it is colinear to $[p_{i+2}, p_{i+1}]$, $i \in \{u,...,v-2\}$. Let α_{i+1} be the angle induced by the rotation. Define $Ang(C) = \Sigma\, \alpha_i$, $i \in \{u+1,...,v-1\}$. If we let the directed line R become successively colinear to $R_u, R_{u+1}, ..., R_{v-1}$ (by rotating R *counterclockwise* about successive p_i, $i \in \{u+1,...,v-1\}$) then we say that R *travels* on C in *clockwise* fashion. Note that if R travels on C from $[p_{v-1}, p_v]$ to $[p_u, p_{u+1}]$ in counterclockwise fashion, then R rotates in clockwise fashion. Also, the value of Ang(C) is independent of the direction in which R travels.

Similarly, given a polygon P, let Ang(P) be the sum of the angles made by the directed line R as it travels in clockwise order once around P until it comes back to the starting edge. In this section of the full paper, we present several lemmas that we use to show the following:

Theorem 4.1. Let P be an L-convex polygon P of $n \geq 3$ vertices of which c are convex. Then $Ang(P) = (c-2)\pi$.
Corollary 4.1.1. Let P be an L-convex polygon P of n vertices of which c are convex. Then the lines of IA(P) can be sorted in O(cn) time.

5. Recognizing Envelopes

Before tackling the recognition problem, we need the following definitions. A line in an arrangement A is said to be *exterior* if it contributes at least 1 line segment to E(A). More generally, a line is said to be *k-exterior* if it contributes exactly k distinct (properly disjoint) line segments to E(A). An arrangement is said to be *exterior* if all the lines of the arrangement are exterior. An arrangement is said to be *k-exterior* if (i) it is exterior, (ii) each line contributes at most k line segments to E(A) and (iii) at least one line is k-exterior. It follows from the definition that an arrangement A is exterior if and only if A = IA(E(A)). This implies that there exists a one-to-one correspondence between exterior arrangements and their envelopes. Thus, if P is a simple polygon but IA(P) is not exterior then P is not an envelope. Hence, for the purposes of the recognition problem, it is sufficient to consider only exterior arrangements. We now introduce a hierarchy of arrangements of lines. Let E_c be the class of exterior arrangements whose envelope contains c convex vertices ($c \in \aleph$). We are now ready to present the first major result of this paper. The proof of theorem 5.1 below is based on corollary 4.1.1 and on the fact that Keil's algorithm [Ke91] for computing the envelope of an arrangement of lines runs in O(n) time when the lines are given sorted in order of slope. Note that the proof of the theorem directly leads to an algorithm.

Theorem 5.1. Let P be an L-convex polygon of n vertices of which c are convex. Then we can recognize in O(cn) time if P is an envelope. i.e. we can recognize in O(n) time if P is the envelope of an arrangement in E_c.

We now introduce the class of sail arrangements. Define a *sail polygon* to be a simple polygon that has exactly three convex vertices. Sail polygons have found applications in the design of efficient algorithms for intersecting convex polygons and triangulating point sets [To85]. We say that an exterior arrangement of lines A is a *sail arrangement* if E(A) is a sail polygon (i.e. if $A \in E_3$). We first show that all sail polygons are envelopes. This implies that given an arbitrary simple polygon P, there is an obvious algorithm to determine in O(n) time whether P is the envelope of a sail arrangement. Furthermore, we observe that the class of sail arrangements is equivalent to another class of arrangements which we call class $\tilde{A}$. We define *class* $\tilde{A}$ to be the set of arrangements A that have the property that the points of S(A) i.e. the intersection points of A, are vertices of E(A) if and only if it they are determined by adjacent lines. It follows from the definition that no two lines of an arrangement $A \in$ class $\tilde{A}$ are parallel. The properties of arrangements in class $\tilde{A}$ (and hence, of sail arrangements) are exploited in section 6, where we determine in O(n) time all the convex vertices of the envelope of a given sail arrangement. The following lemmas, along with theorem 4.1, are used in the proof of theorem 5.2.

Lemma 5.1. Let P be a sail polygon. Then P is an envelope i.e. P = E(IA(P)).

Corollary 5.1.1. We can recognize in O(n) time if a polygon P of n vertices is the envelope of a sail arrangement.

Lemma 5.2. Class $\tilde{A} \subset$ 1–exterior arrangements.

Lemma 5.3. Let A be an arrangement of lines. As we traverse E(A), we meet the lines of A in order by slope if and only if A is in class $\tilde{A}$.

Theorem 5.2. An arrangement of lines A is in class $\tilde{A}$ if and only if A is a sail arrangement.

6. Recognition of the Critical Vertices of a Sail Arrangement

We further classify sail arrangements as k-sail, $0 \le k \le 3$, where k indicates the number of concave chains in the envelope. Owing to the trivial nature of 0-sails (triangles), subsequent discussion omits this subclass. Now we ask the following question. Given an unsorted k-sail arrangement of lines A in general position (i.e. no two lines are parallel and no three lines intersect at a point) $(1 \le k \le 3)$, can the three convex vertices of the resulting envelope be found in linear time (i.e. can we determine which pairs of lines in A realize the three convex vertices of E(A))? It turns out that we can indeed do this in O(n) time. Let the three convex vertices of a sail arrangement be denoted as X, Y and Z. In what follows, we sketch the algorithms that we use to find X, Y and Z in O(n) time for each sail subclass. We later show that we can determine in O(n) time the subclass to which a sail arrangement A belongs.

An edge of E(A) that joins two critical vertices of an arbitrary arrangement of lines A is said to be a *critical edge*. We say that a line of A is a *critical line* if it is colinear to a critical edge. Thus, a k-sail arrangement A has (3 - k) critical edges $(0 \le k \le 3)$. We shall frequently make use of the following simple procedures which have as input a line L of an arrangement $A = \{L_0,...,L_{n-1}\}$ in general position: Compute_Extreme(A,L) which returns the two lines L_i and L_j such that $I(L,L_i)$ and $I(L,L_j)$ are the two extreme points on L; Critical_Line(A,L) which determines whether or not the given line L is critical. We present the following lemma which applies to all sail arrangements and will be useful for subsequent proofs.

Lemma 6.1. Let A be a sail arrangement of n lines and L a line in A. Then L is critical if and only if all of the n-1 points of S(A) on L are extreme.
Corollary 6.1.1. The order of the points of S(A) on a critical line L define a slope ordering of the remaining n-1 lines of A.

We can apply the above results to find the three convex vertices X,Y and Z of a 1-sail arrangement A in O(n) time by applying procedure Compute_Extreme three times and determining in constant time the pairs of lines that form the critical vertices of A.

The algorithm for 2-sails employs the Megiddo technique of prune and search.We find the three convex vertices X,Y and Z of a 2-sail arrangement A in O(n) time with procedure Find_Sail_Tip and algorithm Two_Sail. By definition of a 2-sail arrangement, A has exactly one critical edge. Procedure Find_Sail_Tip accepts the sole critical line L and finds the convex vertex that does not lie on L. This is done by applying the Megiddo technique. Algorithm Two_Sail finds the critical line L (and the two convex vertices that make L critical) and then applies the recursive procedure Find_Sail_Tip, with L as input, to find the third convex vertex of E(A).

We can apply the previous results for 2-sails to obtain a divide and conquer algorithm for 3-sails. Given a 3-sail arrangement of lines, we select an arbitrary line L. From this line, we can obtain the two lines L_i and L_j that determine the extreme intersection points on L. We show that L_i and L_j define two slope ranges that separate the given arrangement of lines into two or three 2-sail arrangements.

Now that we are able to find X, Y and Z given that A is a k-sail $(0 \le k \le 3)$, we observe that we can remove the condition that the value of k be known a priori. The following theorem shows that we need only know that A is a sail arrangement.

Theorem 6.1. Given $A = \{L_0,L_1,...,L_{n-1}\}$, a k-sail arrangement of n lines $(0 \le k \le 3)$, we can determine the value of k and the three convex vertices of A in O(n) time.

7. The Complexity of Constructing the Envelope of a Sail Arrangement

In [CL85], Ching and Lee proved a lower bound of $\Omega(n \log n)$ for the problems of computing the diameter, convex hull and envelope of an arbitrary arrangement of lines. We have introduced and studied sail arrangements with the hopes that definite characterizations of a non-trivial class of arrangements would allow us to beat these lower bounds. Theorem 6.1 implies that given a sail arrangement of n lines, the convex vertices can be determined in $O(n)$ time. This allows us to compute the diameter and convex hull of the arrangement in $O(n)$ time even though there are $O(n^2)$ intersection points. We show in this section, however, that the $\Omega(n \log n)$ lower bound still applies to the envelope construction problem for sail arrangements (by a reduction from Sorting). This implies that the additional information and structure derived from sail arrangements is insufficient for solving the envelope construction problem for sail arrangements in $o(n \log n)$ and ultimately, improves on Ching and Lee's result for arbitrary arrangements.

Theorem 7.1. Given a sail arrangement A of n lines, the problem of computing E(A) is $\Omega(n \log n)$ under the algebraic tree model of computation.

8. Hamiltonian Circuits in Arrangement Graphs

An arrangement of lines A is said to be *hamiltonian* or admit a *hamiltonian circuit* if there exists a simple polygon P_A through all of the points of S(A) such that the edges of P_A are a subset of the set of line segments that are (i) colinear to the lines of A and (ii) bounded by points of S(A). Everett showed that there exists arrangements of lines that are not hamiltonian [Ev91]. Since the property of hamiltonicity does not hold for arbitrary arrangements, the question of whether there are classes of arrangements that are hamiltonian is intriguing. We show in this section that every 1-sail arrangement of lines is hamiltonian and describe an algorithm to produce P_A in polynomial time. Our argument exploits a property of 1-sail arrangements that we establish with the following lemma.

Lemma 8.1. Let $A = \{L_0, L_1, \ldots, L_{n-1}\}$ be a 1-sail arrangement of n lines. Then for each line $L \in A$, the ordering of the n-1 intersection points on L corresponds to the slope ordering of the lines in A.

9. Conclusion

We have introduced a hierarchy of classes of arrangements of lines based on the number of convex vertices of their envelopes. We showed how the properties of envelopes enable us to determine in $O(n)$ time if a polygon of n vertices is the envelope of an arrangement of lines of a given class. In particular, we looked at a class called *sail arrangements* which we prove has properties that enable us to solve a number of problems optimally. Given a sail arrangement A, we can (i) determine the subclass to which it belongs and (ii) find the three convex vertices of E(A) and thus the convex hull, the diameter and the points with minimum or maximum x-coordinate of the arrangement in $O(n)$ time. It is $\Omega(n \log n)$, however, to construct the remainder of the envelope of a sail arrangement. We also showed that 1-sail arrangement graphs admit hamiltonian circuits.

It remains, however, an open problem as to whether or not 2 and 3 -sail arrangement graphs admit hamiltonian circuits. Owing to the well defined geometrical structure of sail arrangements, we conjecture that 2 and 3 -sail arrangements are hamiltonian. The class of sail arrangements is a subclass of the class of 1-exterior arrangements so we are also interested in obtaining similar results for 1-exterior arrangements. Finding the convex vertices of a 1-exterior arrangement of lines in linear time or showing that 1-exterior arrangements are hamiltonian are open questions. In general, for any problem on arrangements of lines which has a known lower bound or no known optimal upper bound, it is interesting to determine if there is a class of arrangements for which the problem can be solved optimally.

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