Strictly Convex Quadrilateralizations of Polygons

Hazel Everett* William Lenhart† Mark Overmars‡ Thomas Shermer§ Jorge Urrutia¶

May 25, 1992

Abstract

A strictly convex quadrilateral is a convex quadrilateral whose angles are all strictly less than 180°. In this paper we show that a polygon on n vertices can always be decomposed into at most $5(n-2)/3$ strictly convex quadrilaterals and that $n-2$ are sometimes necessary. We also show that a polygon on n vertices and h holes can always be decomposed into at most $\lceil 8(n+2h-2)/3 \rceil$ strictly convex quadrilaterals. We give algorithms for constructing the decomposition which run in linear time after triangulation. This problem has applications in mesh generation for finite element methods.

1 Introduction

The problem of triangulating a polygon, that is, decomposing it into non-overlapping triangles, has been thoroughly investigated. It is known that a simple polygon P with n vertices and h holes can always be decomposed into exactly $n+2h-2$ triangles and efficient algorithms for constructing triangulations have been presented [7] [4] [1].

A *convex quadrilateralization* of P is a decomposition of P into non-overlapping convex quadrilaterals. Kahn, Klawe and Kleitman have shown that any *orthogonal* polygon can be decomposed into $n/2+h-1$ convex quadrilaterals [5] and Sack and Lubiw have presented $O(n \log n)$ algorithms for constructing such a quadrilateralization [8] [6].

Triangulating and quadrilateralizing are two examples of the more general operation of decomposing geometric objects into simpler components. These operations are employed in the solution of many geometric problems in areas such as graphics, solid modeling and finite element methods. Typically these applications demand that the simple components satisfy certain properties such as being "well-shaped" and small in number. While some methods are known for generating triangulations satisfying some of these properties (see for example [2]), there are no results for quadrilateralizations. In this paper we consider quadrilateralizations satisfying the "well-shaped" criteria that all quadrilaterals are convex and that no quadrilateral contains three (or more) collinear points.

Formally, let P be a simple polygon on n vertices and h holes. P may contain collinear vertices. A polygon is *strictly convex* if all of its angles are $< 180°$. A *strictly convex quadrilateralization* is a convex quadrilateralization in which all quadrilaterals of the decomposition are strictly convex. Notice that, for example, the polygons in Figure 1 are not considered to be strictly convex quadrilaterals: (a) is not strictly convex and (b) is not a quadrilateral.

If we insist that the vertices of all of the quadrilaterals are also vertices of P then a convex quadrilateralization is not always possible; consider for example a regular pentagon. Therefore, we permit the vertices of the quadrilaterals to be Steiner points, that is, points which are not vertices of P. Notice that once we allow Steiner points the number of quadrilaterals in a decomposition is no longer simply a function of n and h. In this paper we are interested in quadrilateralizations which use a small number of quadrilaterals.

*Dépt. de Mathématiques et Informatique, Université du Québec à Montréal, Montréal, QC, Canada, H3C3P8.

†Dept. of Computer Science, Williams College, Williamstown, MA, USA, 01267.

‡Dept. of Computer Science, Utrecht University, POB 80.089, 350PTB, Utrecht, The Netherlands.

§School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada, V5A 1S6.

¶Dept. of Computer Science, University of Ottawa, Ottawa, ON, Canada, K1N 6N5.

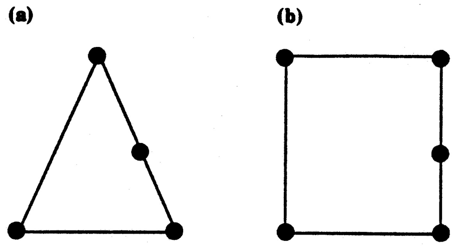

Figure 1: Not strictly convex quadrilaterals

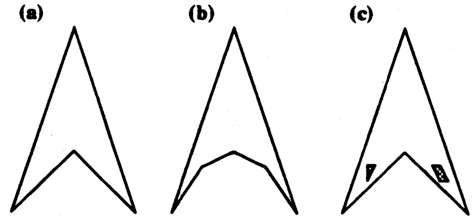

Figure 2: Lower bound examples

We require the following definitions. The ***dual graph*** of a triangulation is the graph containing one vertex for each triangle and an edge between two vertices if the corresponding triangles share a diagonal. If the triangulation does not contain holes then this graph is a tree. A Steiner point p is called ***perturbable*** if it can be moved in such a way that it is a strictly convex vertex in all quadrilaterals that contain it. If p is on the boundary of a subpolygon P' then it is called ***in-perturbable*** (***out-perturbable***) if it can be perturbed to a point in the interior (exterior) of P'.

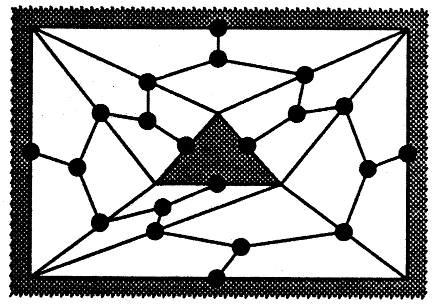

Figure 3: Simple decomposition

2 Polygons with holes

In this section we establish upper bounds on the number of strictly convex quadrilaterals required to quadrilateralize a simple polygon with holes. However, we first present a lower bound on this number.

Lemma 2.1 *Let P be a simple polygon with n vertices and h holes. Then $n-h-2$ strictly convex quadrilaterals are sometimes necessary to quadrilateralize P.*

PROOF Consider the class of spiral polygons containing three convex vertices, examples of which are shown in Figure 2a and b. The midpoints of the edges adjacent to the reflex vertices form a set I of $n-2$ points no two of which can be contained in the same convex subset of the polygon. One can add a small convex hole to any of these examples by placing the hole sufficiently close to the midpoint m of a reflex edge; I is then modified by removing m and adding the midpoints of each of the hole edges (see Figure 2c). After repeated additions of this type, I has $n-h-2$ elements and these polygons therefore require at least that many quadrilaterals. □

Note that this lemma gives a lower bound of $n-2$ quadrilaterals for the case of polygons without holes.

We start our investigation of the upper bound with a simple construction due to de Berg [3] that produces $3n+6h-6$ quadrilaterals. First decompose the polygon into $n+2h-2$ triangles. Decompose each triangle into 3 quadrilaterals using the following approach: place a Steiner point at the midpoint of each diagonal, at the midpoint of each polygon edge, and in the center of each triangle of the triangulation. Connect the Steiner point in the center of each triangle to the three Steiner points on its edges. An example is shown in Figure 3.

The main idea of this section is to improve this algorithm by, rather than quadrilateralizing each triangle separately, first grouping the triangles together into subsets and then quadrilateralizing the subsets.

We start with a general lemma about odd paths in trees. A path is called odd if it contains an odd number of vertices. Note that an isolated vertex is considered an odd path.

Lemma 2.2 *Let T be a binary tree on t vertices, $t \geq 3$. By removing edges, T can be partitioned into at most $\lceil t/3 \rceil$ odd paths.*

PROOF If T is a path then either it is an odd path or it can be partitioned into two odd paths. So assume T is not a path. The proof is by induction on t. Let v be a vertex of degree three none of whose descendants has degree three. The subtree T_v rooted at v is a path of length at least three.

If T_v is an odd path then let $T' = T - T_v$; that is, T' is the tree T after the subtree rooted at v has been removed. The tree T' has $t' \leq t - 3$ vertices and can, by induction, be partitioned into at most $\lceil t'/3 \rceil$ odd paths. These paths together with the path T_v partition T into at most $\lceil t'/3 \rceil + 1 \leq \lceil t/3 \rceil$ odd paths.

If T_v is an even path then let u and w be the children of v. Let T_u and T_w be the subtrees of T rooted at u and w respectively. As T_v has an even number of vertices, one of T_u and T_w has an even number of vertices, say T_u, and the other, say T_w, has an odd number of vertices. Let $T' = T - T_u - T_w$. Note that T' has $t' \leq t - 3$ vertices. By induction, T' can be partitioned into a set Q' of at most $\lceil t'/3 \rceil$ odd paths. One of these paths, Q'_v, contains v as a leaf. Let Q_v be the odd path consisting of Q'_v, T_u and the edge $\{v, u\}$. Now the set of paths $Q' - Q'_v + Q_v + T_w$ partitions T into at most $\lceil t'/3 \rceil - 1 + 2 \leq \lceil t/3 \rceil$ odd paths. □

Theorem 2.3 *A polygon with holes can always be decomposed into at most $\lceil 8(n + 2h - 2)/3 \rceil$ strictly convex quadrilaterals.*

PROOF Consider a triangulation of the polygon into $t = n + 2h - 2$ triangles and its corresponding dual graph. First, place a Steiner point at the midpoint of each polygon edge and each diagonal of the triangulation. Now, let S be a spanning tree of the dual graph. By the previous lemma, S can be partitioned into p odd paths, where $p < \lceil t/3 \rceil$. For each path $Q = q_0, \ldots, q_{2k}$, we quadrilateralize the corresponding set of triangles $t_0, \ldots, t_{2k}$, according to the following scheme. Place a Steiner point at the center of t_{2i}, $0 \leq i \leq k$. See Figure 4. Add an edge from each Steiner point in the center of a triangle to the three Steiner points on the edges of that triangle. In the remaining triangles, add an edge from the vertex common to the two neighboring triangles to the Steiner point on the opposite edge. Although some of the quadrilaterals so created are not strictly convex, they can be made strictly convex by perturbing the Steiner points on the diagonal edges.

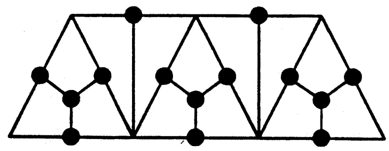

Figure 4: Decomposing an odd path

Color each of the odd paths according to the following scheme: color the first vertex red and then alternate coloring vertices blue and green. Note that each red vertex corresponds to a triangle decomposed into 3 quadrilaterals and that each blue-green pair corresponds to a set of two triangles decomposed into a total of 5 quadrilaterals. There are p red vertices and $(t - p)/2$ blue-green pairs; in total then there are $3p + 5(t - p)/2 = (5t + p)/2$ quadrilaterals. Since p is at most $\lceil t/3 \rceil$, there are at most $(5t + \lceil t/3 \rceil)/2 = \lceil 16t/3 \rceil/2 \leq \lceil 8t/3 \rceil$ quadrilaterals. Since $t = n + 2h - 2$ we have at most $\lceil 8(n + 2h - 2)/3 \rceil$ quadrilaterals. □

3 Polygons without holes

In this section we establish a much improved upper bound of $5(n-2)/3$ on the number of quadrilaterals necessary for polygons without holes. We begin with a series of lemmas about quadrilateralizations of quadrilaterals and pentagons.

Lemma 3.1 *Let F be a quadrilateral with one out-perturbable Steiner point s placed on its boundary. F can be divided into at most three strictly convex quadrilaterals, while placing only one other Steiner point on the boundary of F (on an edge other than the one containing s).*

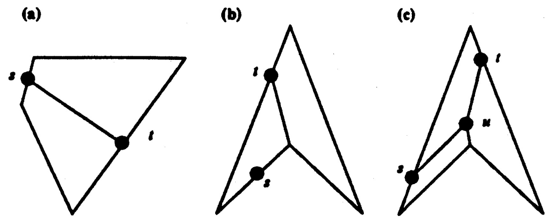

Figure 5: A quadrilateral with one specified Steiner point

PROOF If F is convex, then place a Steiner point t on the edge of F not adjacent to the edge containing s. Connect s and t, as shown in Figure 5a, to divide F into two strictly convex quadrilaterals.

If F is not convex, then s is either on an edge adjacent to the reflex vertex or on one of the two convex edges (edges with both endpoints convex). In the former case, we add a Steiner point t in the kernel of F on the convex edge adjacent to the edge containing s; t is connected to the reflex vertex as shown in Figure 5b. This forms two convex quadrilaterals; out-perturbing s makes them both strictly convex.

If s is on a convex edge, then place Steiner point t on the other convex edge so that s and t are visible. Place another Steiner point u in the kernel of F, on the same side of the chord st as the reflex vertex. Connect as shown in Figure 5c to form three strictly convex quadrilaterals. □

For the following lemmas, we will use the following notation: F will be a pentagon that has been triangulated. This triangulation necessarily consists of three triangles, two of which are ears. Let C denote the center triangle, and L and R be the two ears. The three triangles share a vertex k which is guaranteed to be in the kernel of F. A boundary edge of a triangle is an edge of the triangle on the boundary of F.

Lemma 3.2 *Let F be a pentagon with one Steiner point s placed on its boundary. F can be divided into at most five strictly convex quadrilaterals without placing any other Steiner points on the boundary of F.*

PROOF We consider cases depending on where the Steiner point s lies. The cases are illustrated

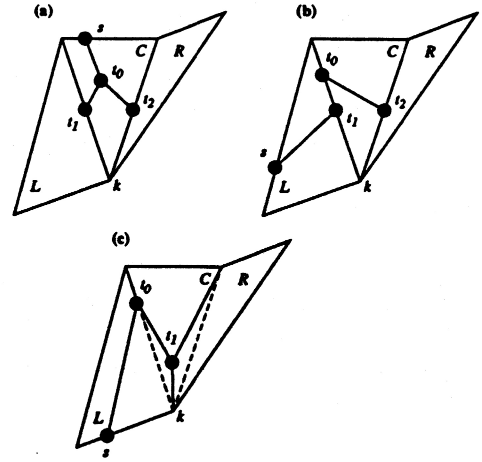

Figure 6: A pentagon with one Steiner point on the boundary

on one particular shape of F but the arguments do not depend on this shape.

Case 1: s is on the boundary edge of C. Add Steiner points t_1, t_2 in the middle of the other two edges of C and t_0 in the center of $\Delta s t_1 t_2$. Connect the vertices of F and the Steiner points as shown in Figure 6a to form five quadrilaterals. These quadrilaterals are (non-strictly) convex, as they are either a triangle of the triangulation, or are part of a "radial decomposition" of C. To make these quadrilaterals strictly convex, move t_1 and t_2 slightly towards t_0.

Case 2: s is on the boundary edge of L (or, symmetrically, R) adjacent to the boundary edge of C.
Add Steiner points t_0 and t_1 to the edge that L shares with C, with t_1 closer to k. Also add a Steiner point t_2 to the edge that R shares with C. Connect as shown in Figure 6b to form five quadrilaterals. Each of these quadrilaterals is convex as it is either a triangle of the triangulation or formed by cutting such a triangle with a chord. To make the quadrilaterals strictly convex, move t_1 towards s and t_2 towards t_0.

Case 3: s is on the boundary edge of L (or, symmetrically, R) adjacent to k.
Add Steiner point t_0 to the edge shared by L and C, and place Steiner point t_1 in the intersection of C with the kernel of F, sufficiently close to k that the line containing s and t_0 does not separate t_1

from k. Connect as shown in Figure 6c to obtain four strictly convex quadrilaterals. □

The proofs of the following two lemmas are similar to that of Lemma 3.2 and are omitted in this extended abstract.

Lemma 3.3 *Let F be a pentagon with one out-perturbable Steiner point s placed on the boundary edge of C, one Steiner point t on the boundary edge of L (symmetrically, R) not adjacent to k, and one Steiner point u on a boundary edge of R (symm., L). F can be divided into at most five strictly convex quadrilaterals without placing any other Steiner points on the boundary of F.*

Lemma 3.4 *Let F be a pentagon with one out-perturbable Steiner point s placed on the boundary edge of C, one Steiner point t on a boundary edge of L, and one Steiner point u on a boundary edge of R. F can be divided into at most six strictly convex quadrilaterals without placing any other Steiner points on the boundary of F.*

Theorem 3.5 *Any n-gon P can be decomposed into at most $5(n-2)/3$ strictly convex quadrilaterals (for $n \geq 4$). Furthermore, this can be done in such a way that every edge of P contains at most one Steiner point.*

PROOF We prove the theorem by induction on n. The basis cases are $n = 4$ and $n = 5$; $n = 4$ is handled as in Lemma 3.1 (avoiding the case where s is out-perturbed) to give three quadrilaterals. $n = 5$ is handled by placing a Steiner point on any edge and applying Lemma 3.2 to give five quadrilaterals. We henceforth assume by induction that the theorem holds for all n smaller than the one we are considering.

First, we triangulate P. Note that there is always a diagonal of the triangulation that cuts off 3-5 triangles (see, e.g. [7], pg. 7); let D be such a diagonal that cuts off the minimum number of triangles. Let the 3-5 triangle fragment thus cut off be called F, and the rest of P be called P'. We consider cases depending on the size of F.

Case 1: F consists of three triangles. Inductively decompose P' into at most $5((n-3)-2)/3$ strictly convex quadrilaterals, in such a manner that D contains either zero or one Steiner points. If D contains a Steiner point, then apply Lemma 3.2 to F with the same Steiner point to yield a decomposition of F into at most five

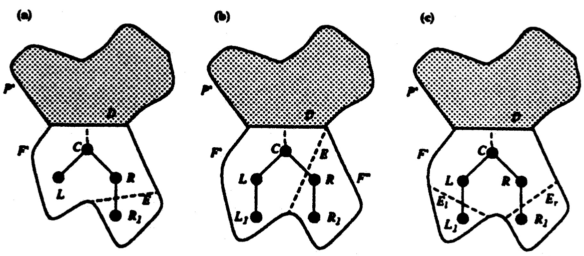

Figure 7: D cuts off 4 or 5 triangles

quadrilaterals. Combining the decompositions for P' and F gives a decomposition of P into at most $5((n-3)-2)/3+5 = 5(n-2)/3$ strictly convex quadrilaterals. If D does not contain a Steiner point, then apply Lemma 3.2 to F with a Steiner point on any edge but D to obtain the same result.

Case 2: F consists of four triangles. As D is minimal, the dual tree of F must have a node C (dual to the triangle containing D) with two children L and R, one of which (wlog R) has a child R_1; this structure is shown in Figure 7a. Let E be the diagonal shared by R and R_1. E may or may not share an endpoint with D. Let F' be the pentagon consisting of the triangles dual to C, L, and R.

Inductively decompose P' into at most $5((n-4)-2)/3$ strictly convex quadrilaterals, in such a manner that D contains either zero or one Steiner points. If D contains no Steiner point, then apply Lemma 3.2 to F' with a Steiner point s on E to give a decomposition of F' into at most five strictly convex quadrilaterals. The triangle dual to R_1, considered with s, is a (not strictly) convex quadrilateral, which we can make strictly convex by perturbing s into F'. We have thus decomposed F into at most six strictly convex quadrilaterals. Combining this with the decomposition for P' gives a decomposition for P with at most $5((n-4)-2)/3+6 < 5(n-2)/3$ quadrilaterals.

If D contains a Steiner point s, then apply Lemma 3.3 to F' with s, a Steiner point t on the edge of L adjacent to D, and a Steiner point u on E to obtain a decomposition of F' into at most five strictly convex quadrilaterals. Again the Steiner point on E is perturbed to make the triangle dual to R_1 a strictly convex quadrilateral, and the same analysis holds.

Case 3: F consists of five triangles.
As D is minimal, the dual tree of F must have a node C corresponding to the triangle containing D with two children L and R, each of which has a child (L_1 and R_1); this structure is shown in Figure 7b and c. Inductively decompose P' into at most $5((n-5)-2)/3$ strictly convex quadrilaterals, in such a manner that D contains either zero or one Steiner points.

If D contains no Steiner point, then let E be the diagonal shared by C and R, F' be the pentagon consisting of the dual triangles to C, L, and L_1, and F'' be the quadrilateral consisting of the dual triangles to R and R_1, as shown in Figure 7b. Apply Lemma 3.2 to F' with a Steiner point s on E to give a decomposition of F' into at most five strictly convex quadrilaterals. Apply Lemma 3.1 to F'' with the Steiner point s to give a decomposition of F'' into at most three strictly convex quadrilaterals (another Steiner point appears on the boundary of F'', but it is not on E and therefore does not cause trouble). Combining the decompositions for P', F', and F'' gives a decomposition of P into at most $5((n-5)-2)/3+8 < 5(n-2)/3$ strictly convex quadrilaterals.

If D contains a Steiner point s, then let E_l be the diagonal shared by L and L_1, E_r be the diagonal shared by R and R_1, and F' be the pentagon consisting of the dual triangles to C, L, and R, as shown in Figure 7c. Apply Lemma 3.4 to F' with s and two other Steiner points, t and u, on E_l and E_r, respectively, to obtain a decomposition of F' into at most six strictly convex quadrilaterals. The triangle dual to R_1, considered with t, is a (not strictly) convex quadrilateral, which we can make strictly convex by perturbing s into F'; we do the same to L_1 and u. Thus, we have decomposed F into at most $6+1+1=8$ quadrilaterals; considered with the decomposition of P' we have obtained a decomposition of P into at most $5((n-5)-2)/3+8 < 5(n-2)/3$ strictly convex quadrilaterals.

Note that we have never placed more than one Steiner point on a polygon edge. □

4 Conclusions

Our results represent a first step in the study of convex quadrilateralizations of simple polygons with holes in the non-axis-parallel case. We have shown upper and lower bounds on the number of convex quadrilaterals required. Our upper bound proofs (for Theorems 2.3 and 3.5) are constructive and can easily be implemented to run in linear time after triangulation of the polygon. However, our bounds are not tight; we suspect that the upper bound can be lowered.

We are also interested in finding fast algorithms to construct more restricted quadrilateralizations, such as requiring the quadrilaterals to have good side-length ratios and no small angles. Such properties are important for the application to finite-element mesh generation.

Acknowledgements

This research was conducted during the 1992 Bellairs Workshop in Computational Geometry. Research of the first author was supported by PAFACC, and research of the first, fourth, and fifth authors was supported by NSERC. The research of the third author was partially supported by the Netherlands Organization for Scientific Research (N.W.O.).

References

[1] R. Bar-Yehuda and R. Grinwald, "Triangulating Polygons with Holes," *Proceedings 2nd Canadian Conf. Comp. Geom.*, 1990, 112-115.

[2] M. Bern, D. Eppstein and J. Gilbert, "Provably Good Mesh Generation," *Proc. 31st Symp. FOCS*, 1990, 231-241.

[3] M. de Berg, private communication, 1992.

[4] B. Chazelle, "Triangulating a Simple Polygon in Linear Time," *Proceedings 31th Symp. FOCS*, 1990, 220-230.

[5] J. Kahn, M. Klawe, and D. Kleitman, "Traditional Galleries Require Fewer Watchmen," *SIAM J. Algebraic and Discrete Methods* 4, 1983, 194-206.

[6] A. Lubiw, "Decomposing Polygonal Regions into Convex Quadrilaterals," *Proceedings 1st ACM Computational Geometry Symposium*, Baltimore, 1985, 97-106.

[7] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, 1987.

[8] J. Sack and G. Toussaint, "A Linear-Time Algorithm for Decomposing Rectilinear Star-Shaped Polygons into Convex Quadrilaterals," *Proc. 19th Allerton Conference on Communication, Control, and Computing*, Monticello, 1981, 21-30.