# Finding All Anchored Squares in a Convex Polygon in Subquadratic Time (Extended Abstract)

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**Abstract**

We present an $O(n \log^2 n)$ method that finds *all* squares inscribed in a convex polygon with $n$ vertices such that at least one corner lies on a vertex of the polygon. We point out that this problem has a lower bound of $\Omega(n \log n)$.

**Keywords:** Convex polygon, inscribed squares, pattern recognition, binary search, lower bound.

## 1 Introduction

Approximating a polygon with a simpler shape is a problem that has received a considerable amount of attention. Finding *inscribed* polygons has applications to pattern recognition, as well as being of theoretical interest in computational geometry. In [2], De Pano, Ke and O'Rourke have described an $O(n^2)$ algorithm for finding the largest inscribed square in a convex polygon $\mathcal{P}$ with $n$ vertices.

The interest in inscribed squares has also been highlighted by Klee in his recent book [6].

Of particular interest are squares that are *anchored*: One corner of the square is located at a vertex of the polygon. While it is relatively easy to find anchored squares in quadratic time, it is nontrivial even to find all squares formed by the $O(n^2)$ diagonals of $\mathcal{P}$ in *subquadratic* time.

## 2 Inscribed Squares and Dual Curves

In the following, we denote the corners of a square by $s_1$, $s_2$, $s_3$ and $s_4$ in counterclockwise order. The vertices of $\mathcal{P}$ are counterclockwise $v_1, \ldots, v_n$, while the edges are $e_1, \ldots, e_n$, where $e_i$ has vertices $v_i$ and $v_{i+1}$.

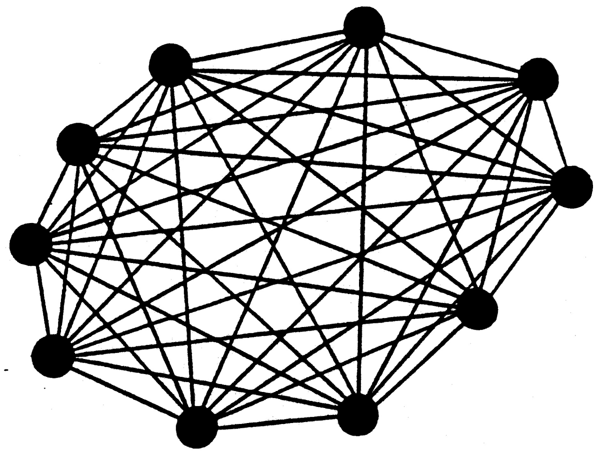

Figure 1: Pattern Recognition: Is there a square among the diagonals of $\mathcal{P}$?

Let $\alpha$ be any point on a convex polygon $\mathcal{P}$. For any point $p$ on $\mathcal{P}$, placing $s_1$ at $\alpha$ and $s_2$ at $p$ positions $s_3$ at the point $\mathcal{R}_\alpha(p)$. Obviously, $\mathcal{R}_\alpha(p)$ is obtained by scaling the distance of $p$ from $\alpha$ by a factor of $\sqrt{2}$ and rotating the resulting point by $\frac{\pi}{4}$ counterclockwise around $\alpha$. Consequently, the locus $\mathcal{R}_\alpha(\mathcal{P})$ of all possible positions of $s_3$ for $s_1$ at $p$ and $s_2$ on $\mathcal{P}$ is a scaled and rotated copy of $\mathcal{P}$, called the *right dual curve* to $\mathcal{P}$.

Similarly, the *left dual curve* $\mathcal{L}_\alpha(\mathcal{P})$ of $\mathcal{P}$ is the locus of all positions of $s_3$ with $s_1$ at $\alpha$ and $s_4$ on $\mathcal{P}$ and obtained by scaling $\mathcal{P}$ by $\sqrt{2}$ and a clockwise rotation of $\frac{\pi}{4}$ around $\alpha$.

**Lemma 2.1** *There is a one-to-one correspondence between squares inscribed in $\mathcal{P}$ anchored at $\alpha$ and points other than $\alpha$ where all three curves $\mathcal{P}$, $\mathcal{R}_\alpha(\mathcal{P})$ and $\mathcal{L}_\alpha(\mathcal{P})$ intersect.*

**Proof.**
Straightforward.
□

Before we describe how to use the dual curves for locating anchored squares, we note the following:

**Theorem 2.2** *Let c be a closed convex curve in the plane and $\alpha$ be some point on c. There is at most one square inscribed in c that is anchored at $\alpha$.*

**Proof.**
Assume there is an anchored square with corners $s_1 = \alpha$, $s_2$, $s_3$ and $s_4$ - see Figure 2. It is not hard to check that it is impossible to place another square with vertices $t_1 = \alpha$, $t_2$, $t_3$ and $t_4$, such that the seven points $\alpha$, $s_2$, $s_3$, $s_4$, $t_2$, $t_3$ and $t_4$ form a convex arrangement.(One of the points $s_2$, $s_4$ will lie inside the square $(\alpha, t_2, t_3, t_4)$ or one of $t_2$, $t_4$ will lie inside the square $(\alpha, s_2, s_3, s_4)$.)
□

We distinguish two kinds of intersections between the dual curves: *Simple* intersections, where an intersection point can be separated from all other intersection points, and

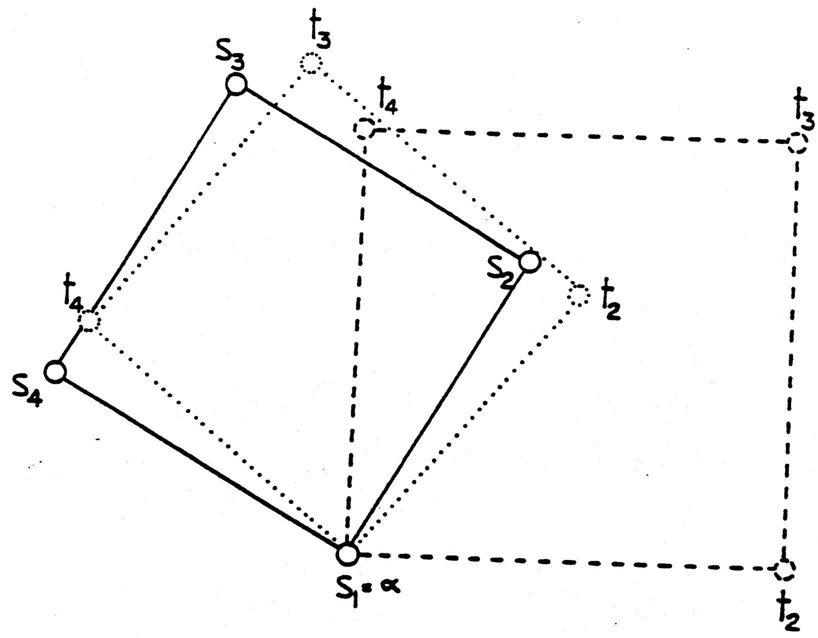


Figure 2: There is at most one inscribed square that is anchored at $\alpha$

*nonsimple* intersections, which consist of a common segment of the polygons $\mathcal{R}_\alpha(\mathcal{P})$ and $\mathcal{L}_\alpha(\mathcal{P})$. Clearly, we get a nonsimple intersection only if there are two edges of $\mathcal{P}$ that enclose an angle of $\frac{\pi}{2}$ and have the same distance from $\alpha$. This property enables us to check all nonsimple intersections in time $O(n \log n)$:

**Algorithm** NONSIMPLE
  **for** each edge $e_i$ of $\mathcal{P}$ **do**
    **if** there is an edge $e_j$ enclosing an angle of $\frac{\pi}{2}$ with $e_i$.
      Determine the unique point $p_i$ on $\mathcal{P}$ that has the same positive distance from $e_i$ and $e_j$.
      Check whether $\mathcal{R}_{p_i}(e_i)$ and $\mathcal{L}_{p_i}(e_j)$ intersect on $\mathcal{P}$.
  **return**
**End of** NONSIMPLE.

Note that NONSIMPLE detects even those inscribed squares with corresponding nonsimple intersections that are not anchored at a vertex of the polygon $\mathcal{P}$.

## 3 Simple Intersections

We will now discuss the problem of detecting inscribed squares with corresponding simple intersection of the dual curves.

Assume $\alpha$ is an anchor point for which there exists an inscribed square with a simple intersection point $t$; see Figure 3. (The shaded areas indicate areas that cannot contain any part of $\mathcal{R}_\alpha(\mathcal{P})$, or $\mathcal{L}_\alpha(\mathcal{P})$ rsp., because of convexity.) We see that as a consequence of convexity of $\mathcal{P}$, $\mathcal{R}_\alpha(\mathcal{P})$ and $\mathcal{L}_\alpha(\mathcal{P})$, any other intersection point $t'$ of the dual curves must satisfy $|\angle(t', \alpha, t)| > \frac{\pi}{4}$. Furthermore, for any two other such intersection points $t'$ and $t''$,

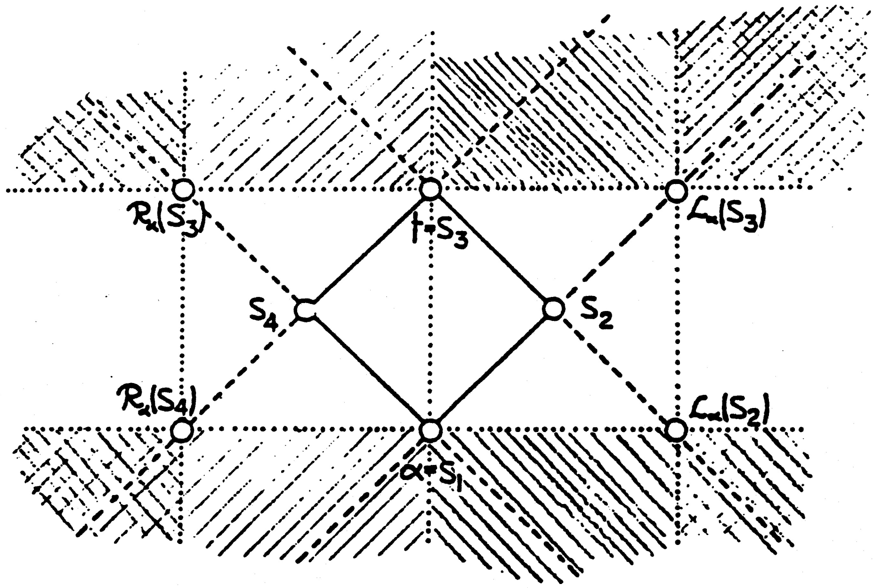


**Figure 3: The situation for a square with a simple intersection**

we get $|\angle(t', \alpha, t'')| < \frac{\pi}{4}$. Finally, we see that the two dual curves cross each other at $t$.

This implies the following algorithm:

**Algorithm SQUARE**

use NONSIMPLE to detect all nonsimple intersections.
for each vertex $v_i$ of $\mathcal{P}$ do
  if no nonsimple intersection $t'$ for anchor point $v_i$,
    use binary search to determine a simple intersection point $t'$.
  if intersection point $t'$ does not yield square,
    Use binary search on $\{t \in \mathcal{L}_\alpha(\mathcal{P}) \mid \frac{\pi}{4} < |\angle(t, \alpha, t')|\}$ to
    detect any simple intersection point $t$ corresponding to
    an inscribed square.
return all squares $Q_i$.

**End of SQUARE.**

For the binary searches, we use the following idea:

Consider a ray from $\alpha$ through a vertex of $\mathcal{L}_\alpha(\mathcal{P})$. In time $O(\log n)$, determine the (unique) intersection point $q \neq \alpha$ with $\mathcal{R}_\alpha(\mathcal{P})$. If $q$ lies outside $\mathcal{L}_\alpha(\mathcal{P})$, an intersection must lie clockwise from $q$, as seen from $\alpha$. If $q$ lies inside $\mathcal{L}_\alpha(\mathcal{P})$, an intersection must lie counterclockwise from $q$, as seen from $\alpha$. When we are left with an edge as our search interval, we can calculate the intersection point.

Using this binary search on the vertices of $\mathcal{L}_\alpha(\mathcal{P})$, we get an overall complexity of $O(n \log^2 n)$.

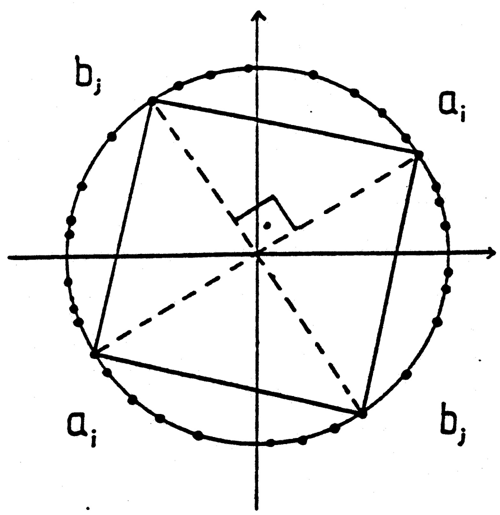


Figure 4: An anchored square implies $a_i = b_j$

# 4 A Lower Bound

We point out that a method by R.L.DRYSDALE and J.W.JAROMCZYK (cf. [3]) implies a lower bound of $\Omega(n \log n)$:

**Theorem 4.1** *Determining whether there is square inscribed in a convex polygon that is anchored at a vertex has a lower bound of $\Omega(n \log n)$ in the algebraic computation tree model.*

**Proof.**

Reduce the set disjointness problem to the square problem: For two given sets $\{a_i \mid i = 1, \ldots, m\}$ and $\{b_j \mid j = 1, \ldots, m\}$ of positive integers, take a sufficiently large integer $M$.

Map $a_i$ onto the angles $\frac{\pi a_i}{2M}$ and $\frac{\pi a_i}{2M} + \pi$, while $b_j$ gets mapped onto $\frac{\pi b_j}{2M} + \frac{\pi}{2}$ and $\frac{\pi b_j}{2M} + \frac{3\pi}{2}$. These values correspond to arcs on the unit circle, hence to a set of points. (The points for the $a_i$ lie in the first and third quadrant, the ones for $b_j$ in the second and fourth quadrant.)

Now it is not hard to see that every square inscribed in the unit circle has diagonals intersecting at the center of the circle. Knowing all anchored squares inscribed in the constructed polygon includes knowing whether there is one with a diagonal of length 2. There is such a square if and only if there is some $a_i = b_j$.

□

# 5 Conclusion

We have presented an $O(n \log^2 n)$ algorithm for determining all anchored squares inscribed in a convex polygon with $n$ vertices. Since there is a lower bound of $\Omega(n \log n)$, it would be particularly nice to improve our algorithm to $O(n \log n)$. This might be possible with a more sophisticated approach for locating simple intersections of the two dual curves.

Another interesting question is to give a subquadratic algorithm for finding maximal inscribed squares that are not anchored, i.e. that have no corners on vertices. This would improve the method of [2] for finding maximal inscribed squares to quadratic running time. It remains an open question whether there can be a superlinear number of maximal squares of this type.

Our method can be immediately generalized for finding inscribed rectangles with a given ratio of sides. Other quadrangles make it necessary to give some more specifications - we have omitted a detailed discussion at this point. It is not true for general convex quadrangles that there can only be one similar inscribed copy anchored at a vertex. (Theorem 2 cannot even be generalized to rhombi, i.e. quadrangles with four equal sides.)

We do conjecture, however, that the overall number of anchored quadrangles will still be linear.

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