EXTENSION OF THE NOTION OF MAP FOR THE REPRESENTATION OF THE TOPOLOGY OF CELLULAR COMPLEXES

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1 Introduction

During the last years, many works have dealt with the definition of combinatorial models for representing the topology of *subdivisions of topological spaces* (a subdivision is a partition of the space into cells of dimensions 0, 1, 2, 3 ..., i.e. into vertices, edges, faces, volumes ...). Handling subdivisions is required for many problems (for instance for computing Voronoï diagrams, and more generally for geometric modelling). These combinatorial models present many interests (cf. [10]). For instance, it is often possible to easily deduce data structures from these combinatorial models. Moreover, it is often possible to compute important topological properties on these combinatorial models (orientability for instance). In fact, efficient combinatorial models are very important for the conception of efficient basic algorithms (i.e. such models may be used in order to reduce the complexity of geometric algorithms [4], [10]).

We may distinguish between *simplicial or cellular* combinatorial models. Models (cf. [8] for instance) have been defined for representing the topology of *simplicial complexes*, i.e. (informally) collections of simplices (a 0-dimensional simplex, or 0-simplex, is a vertex, a 1-simplex is an edge, a 2-simplex is a triangle, a 3-simplex is a tetrahedron ...). Other models are *cellular* ones : it is possible to represent collections of any cells (vertices, edges, faces, volumes .... : cf. for instance [1], [3], [6], [17], [14]), i.e. it is possible to represent *cellular complexes*. For each type of model (simplicial or cellular), we may distinguish between *manifold* and *non-manifold* combinatorial models. Many models have been defined for modeling the topology of subdivisions of *manifolds* (manifold modeling ; cf. for instance [2], [3], [4], [5], [10], [11], [13], [16], [17]). Informally (and with a constructive point of view), an n-dimensional "manifold" object may be constructed by *"joining" at most two n-dimensional cells along an (n-1)-dimensional cell which bounds them* (for instance, joining 2 faces along an edge, 2 volumes along an incident face ...). Other models have been defined for representing the topology of *"non-manifold"* geometric objects. Examples of non-manifold objects are : an object composed by two volumes which are only adjacent through a vertex, a volumic object with dangling edges and faces, an object with an incomplete boundary ... (cf. for instance [8], [14], [15], [18]).

Many important problems are not solved, concerning combinatorial models (in particular cellular models). For instance, what are the "geometric objects" whose topology is described by cellular models (for instance, a cell is possibly not a simplex. But what is a cell ?) ? What are the relations between the different combinatorial cellular models (for instance between *incidence graphs* and *ordered models* : cf. [4]) ?

We have studied these problems in the following way :

– we have defined a combinatorial simplicial model, slightly generalized compared to classical simplicial (it is possible to define objects composed by simplices, such that an i-dimensional simplex is possibly not incident to lower-dimensional simplices, i.e. the boundary of a simplex is possibly incomplete) ;

– we have constructed a hierarchy of simplicial models, by restricting the modeling domain (i.e. for instance, we deduce models which allow the representation of simplicial manifolds from models which allow the representation of simplicial complexes ; this construction is a classical one, and produces well-known models).

– we have defined cellular models as simplicial models to which a structuration is added. In other words, a combinatorial cellular complex is defined as a combinatorial simplicial complex structured into cells (this idea may be deduced from [4]). An i-dimensional cell is an i-dimensional simplicial manifold which satisfies some properties. In particular, each vertex of the simplicial manifold which defines the cell is associated with a number between 0 and i, and a unique vertex is associated with i (cf. [7] for more details). We have defined a combinatorial cellular model, whose definition is very close (in its principle) to the definition of the simplicial model presented above.

– as for the simplicial case, we have constructed a hierarchy of cellular models, by restricting the modeling domain. It is easy to show that most combinatorial cellular models used in geometric modeling are equivalent to one of these combinatorial models (manifold or non-manifold models).

We present in this paper the second part of this study, i.e. the definition of the basic cellular model (*n-dimensional chains of maps*), and some models deduced from this model. *Chains of maps* are an extension of *n-dimensional generalized maps* (or *n-G-maps* : cf. [11]). It is possible to represent the topology of *non-manifold cellular* objects (i.e. *cellular complexes*). More precisely, an n-dimensional chain is composed by 0-, 1-, ... n-dimensional cells (i-cells are represented by an i-G-map G^i). To each i-cell are associated the lower-dimensional j-cells which define its *boundary* (by applications σ^i_j, $0 \leq j < i$, defined on G^i). A unique low-level operation is needed for handling chains of maps (contrary to other models, which need many basic operations : cf. [18] for instance), simplifying thus the construction of chains. Other models, corresponding to particular classes of chains, may be easily deduced from the definition of chains (cf. section 4), for instance "closed" n-dimensional chains. The definition of n-dimensional generalized maps is reminded in section 2. Chains of maps are defined in section 3. Particular cases of chains are studied in section 4. We conclude in section 5.

2 N-dimensional generalized maps

In this section, the notion of the n-dimensional generalized map (or n-G-map) is reminded ([11]). In order to simplify, we give a *recursive* definition of n-G-maps. Definition of n-G-maps is based on a unique type of elements (*darts*), on which elementary functions act. These functions are *involutions* (an involution α is a bijection, such that α^2 = identity) : cf. Fig. 1. A unique type of object (*orbits*), defined using darts and involutions, unifies the definitions of *cells, connected components...* Important topological properties, as orientability for instance, may be computed using the notion of orbit. A unique basic operation (*sewing*) is needed for constructing any n-G-map, simplifying thus the handling of n-G-maps.

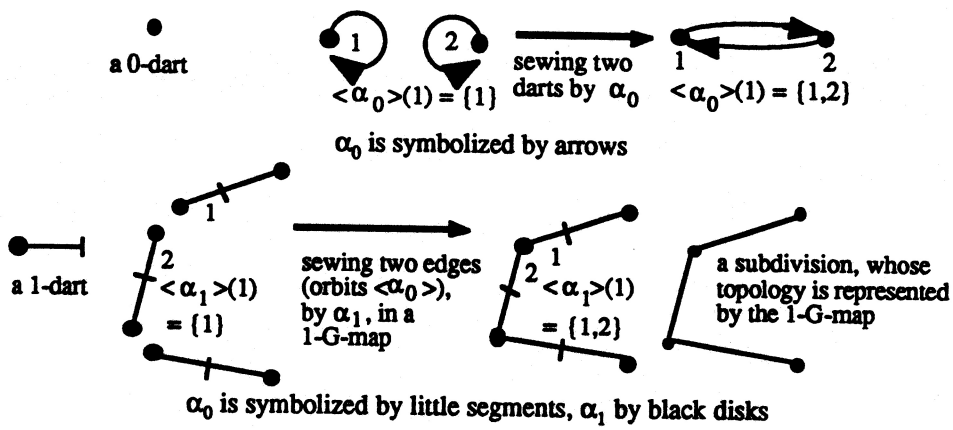
A *(-1)-dimensional generalized map* (or *(-1)-G-map*) is defined by G = (B), where B is a finite set of *darts*. An *n-dimensional generalized map* (or *n-G-map*) is defined by an (n+2)-tuple $G = (B,\alpha_0,...,\alpha_n)$, with :

– $(B,\alpha_0,...,\alpha_{n-1})$ is an (n-1)-G-map ;

– α_n is an involution on B, such that, for any i ($0 \leq i \leq n-2$), $\alpha_i\alpha_n$ is an involution (notations : $\alpha(b) = b\alpha$, and $(\alpha' \circ \alpha)(b) = b\alpha\alpha'$).

The *orbit* $<\alpha_{i0}, \alpha_{i2}, ...\alpha_{ip}>(b)$, incident to dart b, is the set S_b of all darts which "can be reached", starting from b, by successive applications of involutions α_{ij} ($0 \leq j \leq p$; more precisely, this orbit is the p-G-map $(S_b,\alpha'_{i0}, \alpha'_{i2}, ...\alpha'_{ip})$, where α'_{ij}, is the restriction of α_{ij}, to S_b, $0 \leq j \leq p$). The *i-dimensional cell* incident to dart b (or *(i,n)-cell* of n-G-map G, incident to dart b) is defined by $<\alpha_0, ... \alpha_{i-1}, \alpha_{i+1}, ... \alpha_n>(b)$. The *connected component* incident to dart b is defined by $<\alpha_0, \alpha_1, ...\alpha_n>(b)$ (cf. Fig. 1). The *boundaries* of a connected component may be easily defined using the darts of the connected component which are invariant by α_i (for i between 0 and n : cf. Fig. 1 and [11]).

Any n-G-map may be constructed by successive applications of a unique basic operation : *sewing*. Examples of sewing are shown in Fig. 1 : 2-dimensional sewing consists in sewing at most two faces (orbits $<\alpha_0, \alpha_1>$) by α_2, along edges which bound these faces (i.e. orbits $<\alpha_0>$) ; 3-dimensional sewing consists in sewing at most two volumes (orbits $<\alpha_0, \alpha_1, \alpha_2>$) by α_3, along faces which bound these volumes (i.e. orbits $<\alpha_0, \alpha_1>$). More generally, n-dimensional sewing consists in joining at most two n-cells (orbits $<\alpha_0, \alpha_1, ...\alpha_{n-1}>$) along (n-1)-cells which bound these n-cells (i.e. orbits $<\alpha_0, \alpha_1, ...\alpha_{n-2}>$). Other notions concerning n-G-maps (duality, orientability ...) are presented in [11]. A (partial) comparison between n-G-maps and other combinatorial models is presented in [12].



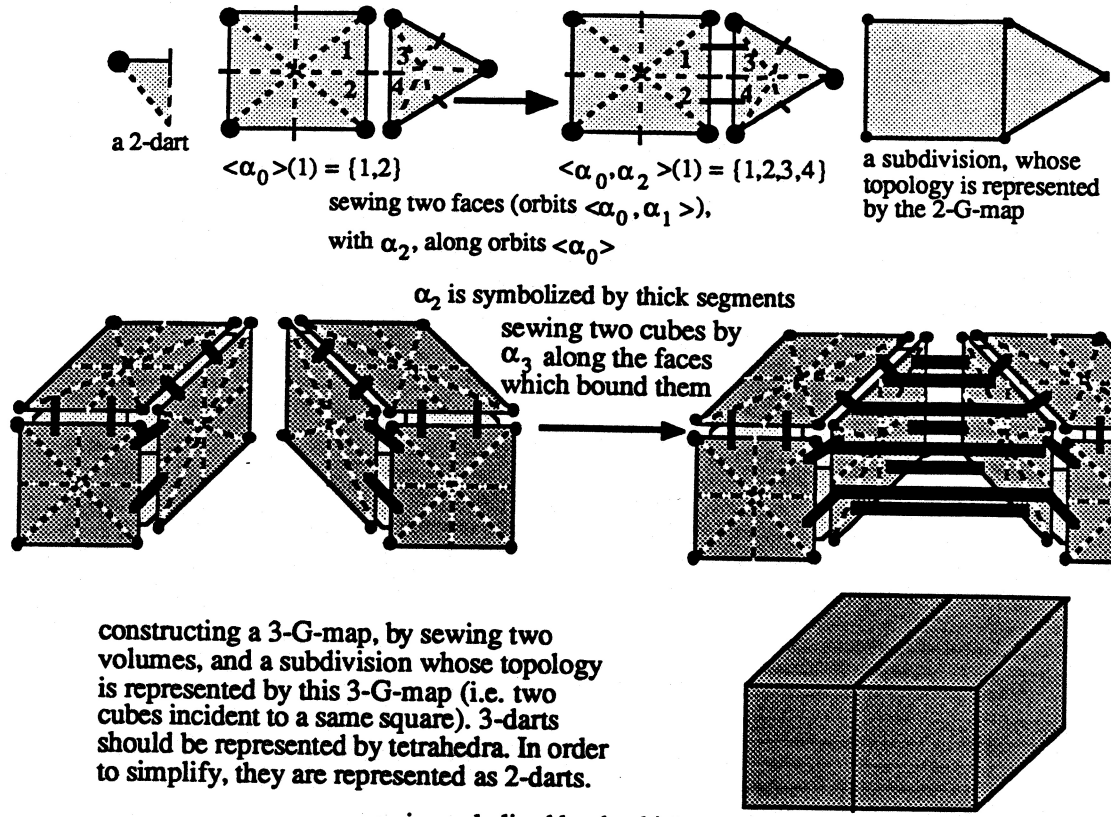


Figure 1 : 0-, 1-, 2- and 3-G-maps (the graphical represention of darts depends on the dimension of the generalized map ; this graphical representation is related to the fact that cells are in fact defined as structured simplical manifolds). The boundary of the 2-G-map (corresponding to the boundary of the associated subdivision) is composed by edges incident to the darts which are invariant by α_2 (i.e. darts b, such that $b\alpha_2 = b$) ; similarly, the boundary of the 3-G-map (corresponding to the boundary of the associated subdivision) is composed by faces which are incident to the darts invariant by α_3.

3 N-dimensional chains

N-dimensional chains of maps (or n-chains) are a combinatorial model, which may be used for representing collections of cells and incidence relations between these cells. In fact, it is possible to represent, using n-chains, the topology of non-manifold cellular objects with (maybe) incomplete boundaries. Chains of maps are presented here in a constructive way (by recursion on the dimension of the chain), i.e. we show how to construct chains by "adding" cells (cf. also the construction of CW-complexes in [9]). More precisely, an n-chain is composed by i-dimensional cells (for i between 0 and n). Each i-dimensional cell is an (i,i)-cell of an i-G-map $G^i = (B^i, \alpha_0^i, \alpha_1^i, \ldots \alpha_i^i)$, such that α_i^i is the identity on B^i. In fact, we will consider that the i-G-map defines only this (i,i)-cell, and not its boundary (this is not strictly the interpretation of n-G-maps given in the previous section. N-G-maps as defined in section 2 may be used for the representation of cellular manifold objects. They correspond in fact to a particular case of n-chains, as claimed in section 4). To each i-cell are associated the lower-dimensional j-cells which define its boundary, by applications σ_j^i, $0 \leq j < i$, defined on the darts of G^i.

– a *0-chain* C is defined by $C = (G^0)$, where $G^0 = (B^0, \alpha_0^0)$ is a 0-G-map, such that α_0^0 = identity on B^0. Each (0,0)-cell of G^0 is a *0-dimensional cell* of the chain (i.e. a *vertex* or a *0-cell* : cf. Fig. 2) ;

– a *1-chain* C is defined by a 3-tuple $C = (G^0, G^1, \sigma_0^1)$, with :

- (G^0) is a 0-chain ;
- $G^1 = (B^1, \alpha_0^1, \alpha_1^1)$ is a 1-G-map, such that α_1^1 = identity on B^1. Each (1,1)-cell of G^1 is a *1-cell* (i.e. an *edge*) of the chain ;
- $\sigma_0^1 : B^1 \rightarrow B^0 \cup \{\varepsilon\}$ is an application which associates to *each vertex of* G^1 (i.e. (0,1)-cell of G^1), either a *vertex of the chain* (i.e. a (0,0)-cell of G^0 ; in this case, for each dart b of the (0,1)-cell, $b\sigma_0^1 \in B^0$), either *nothing* (in this case, for each dart b of the (0,1)-cell, $b\sigma_0^1 = \varepsilon$; the boundary of the 1-cell is then *incomplete*). Intuitively, σ_0^1 defines the boundaries of the edges of the chain (if they exit), i.e. σ_0^1 *may* associates a vertex to each extremity of each edge of the chain : cf. Fig. 2 ;

– a *2-chain* C is defined by $C = (G^0,G^1,G^2,\sigma_0^1,\sigma_0^2,\sigma_1^2)$, with :

• (G^0,G^1,σ_0^1) is a 1-chain ;

• $G^2 = (B^2,\alpha_0^2,\alpha_1^2,\alpha_2^2)$ is a 2-G-map, such that α_2^2 = identity on B^2 ; each (2,2)-cell of G^2 is a *2-cell* of the chain (i.e. a *face*) ;

• $\sigma_0^2 : B^2 \rightarrow B^0 \cup \{\varepsilon\}$ is an application which associates to *each vertex of* G^2 (i.e. to each (0,2)-cell of G^2), either a *vertex of the chain* (i.e. a (0,0)-cell of G^0), either *nothing* ;

• $\sigma_1^2 : B^2 \rightarrow B^1 \cup \{\varepsilon\}$ is an application which associates to *each edge of* G^2 (i.e. to each (1,2)-cell of G^2), either an *edge of the chain* (i.e. a (1,1)-cell of G^1), either *nothing*. Moreover, σ_1^2 and σ_0^2 are such that, for any dart b of B^2, if $b\sigma_1^2 \neq \varepsilon$, $b\sigma_0^2 = b\sigma_1^2\sigma_0^1$ (i.e. if an edge of the chain is associated with an edge of G^2, these edges have a same boundary : cf. Fig. 2) ;

– a *3-chain* C is defined by $C = (G^0,G^1,G^2,G^3,\sigma_0^1,\sigma_0^2,\sigma_1^2,\sigma_0^3,\sigma_1^3,\sigma_2^3)$, with :

• $(G^0,G^1,G^2,\sigma_0^1,\sigma_0^2,\sigma_1^2)$ is a 2-chain,

• $G^3 = (B^3,\alpha_0^3,\alpha_1^3,\alpha_2^3,\alpha_3^3)$ is a 3-G-map, such that α_3^3 = identity on B^3 ; each (3,3)-cell of G^3 is a *3-cell of the chain* (i.e. a *volume*) ;

• $\sigma_0^3 : B^3 \rightarrow B^0 \cup \{\varepsilon\}$ (resp. $\sigma_1^3 : B^3 \rightarrow B^1 \cup \{\varepsilon\}$, $\sigma_2^3 : B^3 \rightarrow B^2 \cup \{\varepsilon\}$) is an application which associates to *each vertex (resp. edge, face) of* G^3 (i.e. to each (0,3)-cell (resp. (1,3)-cell, (2,3)-cell) of G^3), either *a vertex (resp. edge, face) of the chain* (i.e. a (0,0)-cell of G^0 (resp. a (1,1)-cell of G^1, a (2,2)-cell of G^2)), either *nothing* (in this case, the boundary of the 3-cell of the chain is incomplete). Applications σ_i^3 $(0 \leq i \leq 2)$ are such that, if an edge (face) of the chain is associated with an edge (face) of the 3-cell, these edges (faces) have a same boundary (cf. definition of n-chains below).

This may be easily generalized for higher dimensions, and an *n-chain* C is defined by $C = (G^0,\ldots,G^n,\sigma_0^1,\ldots,\sigma_0^n,\ \ldots\ \sigma_{n-1}^n)$, with :

– $(G^0,\ldots,G^{n-1},\sigma_0^1,\ldots,\sigma_0^{n-1},\ldots,\sigma_{n-2}^{n-1})$ is an *(n-1)-chain*, which defines the 0-, 1-, ... (n-1)-cells, and the relations between these cells ;

– $G^n = (B^n,\alpha_0^n,\ldots,\alpha_{n-1}^n,\alpha_n^n)$ is an n-G-map, such that α_n^n = identity on B^n, whose (n,n)-cells define the *n-cells* of the chain ;

– for each i between 0 and n-1, $\sigma_i^n : B^n \rightarrow B^i \cup \{\varepsilon\}$ is an application which associates to each *(i,n)-cell* of G^n, either an *i-cell* of the chain (in this case, for each dart b of the (i,n)-cell, $b\sigma_i^n \in B^i$), either *nothing* (in this case, for each dart b of the (i,n)-cell, $b\sigma_i^n = \varepsilon$; the boundary of the n-cell is then incomplete).

These applications satisfy the following properties :

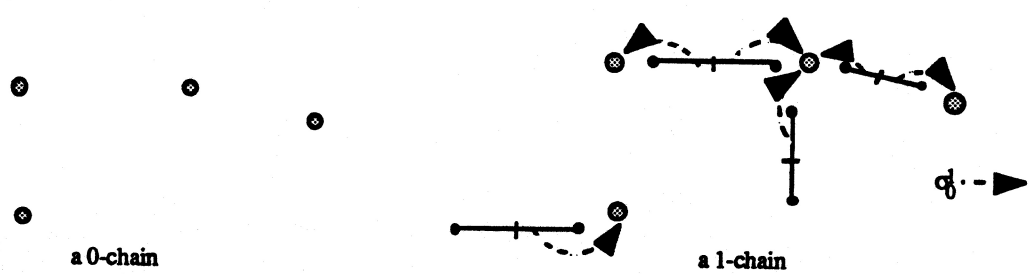
– (C1) $\forall\ b \in B^n$, and for each k, $0 \leq k \leq i-1$, $b\alpha_k^n\sigma_i^n = b\sigma_i^n\alpha_k^i$, or $b\alpha_k^n\sigma_i^n = b\sigma_i^n$,

– (C2) $\forall\ b \in B^n$, and for each k, $i+1 \leq k \leq n-1$, $b\alpha_k^n\sigma_i^n = b\sigma_i^n$,

– (C3) $\forall\ b \in B^n$, and for any i, j, such that $0 \leq j < i \leq n-1$, $b\sigma_i^n \neq \varepsilon \Rightarrow b\sigma_j^n = b\sigma_i^n\sigma_j^i$.

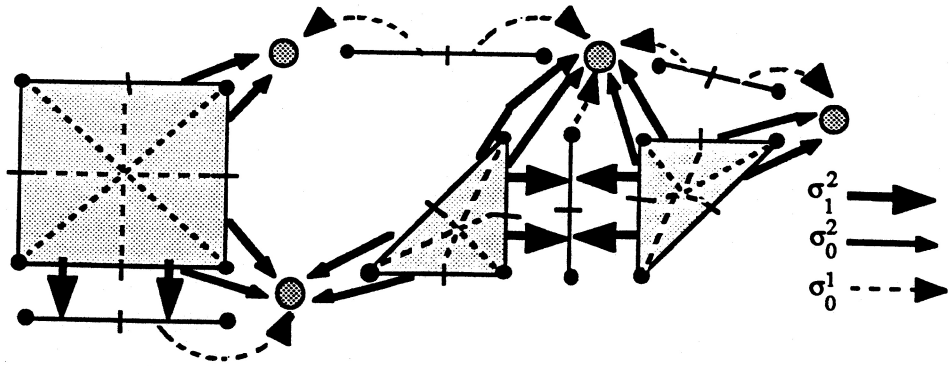
For instance, (C3) means that, if an (i,n)-cell is associated with an i-cell, these two cells have a same boundary.

It is clear that it is possible to represent, using chains of maps, the topology of non-manifold cellular objects (cf. Fig. 2). Many notions, classical in algebraic topology, may be easily defined on n-chains, for instance notions of connected component, skeleton, ... For instance, it is clear that the *boundary* of a cell may be easily defined (similar for the *star* of a cell, i.e. the set of cells whose boundaries contain the cell) : cf. Fig. 2. These different notions are unified through a same formalism, which is the notion of *(i,n)-chain* (this notion is similar to the notion of orbit defined for n-G-maps). An (i,n)-chain $C = (G^i,\ldots,G^n,\sigma_i^{i+1},\sigma_i^{i+2},\sigma_{i+1}^{i+2},\ldots,\sigma_i^n,\ldots\ \sigma_{n-1}^n)$ is defined as an n-chain (just replace dimension 0 by dimension i in the definition of n-chains above ; an n-chain is in fact a (0,n)-chain : cf. [7]). It is possible to construct any n-chain by a unique operation (*"adding"* operation : cf. Fig. 2). In fact, this operation may be defined as the composition of two operations : an operation for adding an i-cell without boundary, and an operation for joining a (j,i)-cell of an i-cell to a j-cell. It is thus possible to gradually construct the boundary of the i-cell (in an interactive process, for instance).

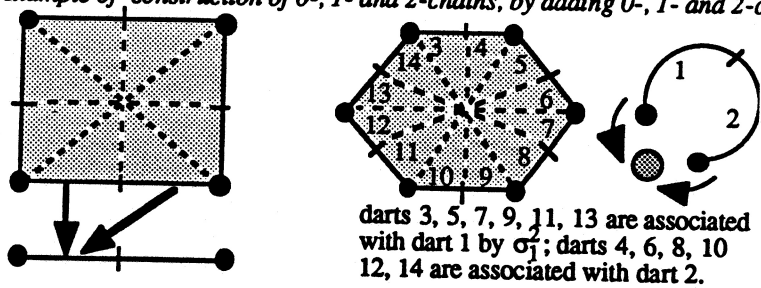


a 0-chain

a 1-chain



an example of construction of 0-, 1- and 2-chains, by adding 0-, 1- and 2-cells.



darts 3, 5, 7, 9, 11, 13 are associated with dart 1 by σ_1^2; darts 4, 6, 8, 10 12, 14 are associated with dart 2.

Figure 2 : examples of 2-chains : notice that σ_j^i are possibly not isomorphisms (left) ; right, a cellular complex composed by a vertex, an edge and a face, such that the face is incident 6 times to the edge. Notice that if the degree of the face is 4 (i.e. it is incident to 4 darts), this cellular complex is an (abstract) projective plane.

4 A hierarchy of combinatorial cellular models

Many particular cases of the notion of n-chain may be defined, for representing sub-classes of cellular complexes ([7]). For instance, the topology of cellular complexes, in which each cell has a complete boundary, may be represented by *"closed" n-chains*. The definition of this notion is similar to the n-chain definition. A closed n-chain is composed by i-dimensional cells ($0 \leq i \leq n$). Each i-cell is an (i,i)-cell of an i-G-map G^i. To each i-cell are associated the (i-1)-cells of its *boundary*, by an application σ^i, defined on the darts of G^i.

More precisely, a *closed 0-chain* C is defined by $C = (G^0)$, where $G^0 = (B^0, \alpha_0^0)$ is a 0-G-map, such that α_0^0 = identity on B^0. A *closed n-chain* C is defined by $C = (G^0,...,G^n,\sigma^1,...,\sigma^n)$, with :

- $(G^0,...,G^{n-1},\sigma^1,...,\sigma^{n-1})$ is a closed (n-1)-chain ;
- $G^n = (B^n,\alpha_0^n,...,\alpha_{n-1}^n,\alpha_n^n)$ is an n-G-map, such that α_n^n = identity on B^n ; each (n,n)-cell of G^n is an *n-cell* ;
- $\sigma^n : B^n \rightarrow B^{n-1}$ is an application which associates an *(n-1)-cell* of the chain (i.e. an (n-1,n-1)-cell of G^{n-1}) to each *(n-1,n)-cell of G^n*. Moreover, σ^n is such that an *i-cell of the chain* ($0 \leq i \leq n-2$) is associated to *each (i,n)-cell of G^n*, by applying $\sigma^n...\sigma^{i+1}$, i.e. σ^n satisfies the following properties :
  - $\forall\, b \in B^n$, $\forall\, k$ ($0 \leq k \leq n-2$), $b\alpha_k^n\sigma^n = b\sigma^n\alpha_k^{n-1}$ or $b\alpha_k^n\sigma^n = b\sigma^n$;
  - $\forall\, b \in B^n$, $b\alpha_{n-1}^n\sigma^n\sigma^{n-1} = b\sigma^n\sigma^{n-1}$.

These two properties correspond to properties (C1), (C2) and (C3) defined for n-chains (cf. section 3). Notions defined for n-chains (in section 3) may be defined for closed n-chains in a similar way : *boundary* of a cell, *star* of a cell, *connected component*, *skeleton* ... These notions are also defined through a unique notion of *closed (i,n)-chain*. Any closed n-chain may be constructed by applying a unique operation (*adding* operation). Notice that a cell may be added if its boundary has been previously constructed. Adding cells in a closed n-chain is thus more constraining than adding cells in an n-chain.

It is also possible to define other combinatorial models which are particular cases of the notion of n-chains. For instance, cellular manifolds (resp. oriented cellular manifolds without boundaries) may be represented by n-dimensional generalized maps (resp. n-dimensional oriented maps) : cf. [11]. It is then possible to define a hierarchy of combinatorial models (from n-chains to n-maps), each one corresponding to a particular sub-class of cellular complexes (cf. [7]).

5 Conclusion

The topology of non-manifold cellular objects (cellular complexes), with incomplete (resp. complete) boundaries, may be represented by n-chains (resp. closed n-chains). These two notions are based on :

- a unique type of basic elements : darts ;

– two types of functions which act on darts :

• *intra-cellular* operators (which are involutions), which define the cells of an n-chain (closed or not). All notions (associated to these cells) are defined through a unique notion of *orbit*. Any cell may be constructed by a unique *sewing* operation.

• *inter-cellular* operators (which are applications), which define the relations between cells of an n-chain (closed or not). Most notions associated to (closed) n-chains are defined through a unique notion of *(i,n)-chain*. Any (closed) n-chain may be constructed by a unique *adding* operation.

Chains of maps are in fact a generalization of ordered models (cf. [4] and [12]) for the representation of cellular complexes. Contrary to most combinatorial cellular models used in geometric modelling, the relations between chains of maps (and the deduced models : closed chains of maps, n-dimensional generalized maps...) and algebraic topology are clearly established (informally, n-chains describe simplicial complexes structured into cellular complexes : cf. [7]).

We are studying the following developments :

– the definition of sub-classes of n-chains, for representing particular sub-classes of cellular complexes (for instance, closed n-chains are such a particular sub-class) ;

– the *embedding* of n-chains (in E^n, for instance) ;

– the implementation and experimentation of the basic notions presented here ;

– problems related to particular applications, for instance interactive geometric modeling and discrete topology.

Acknowledgments : This study has been realized with the financial help of the French P.R.C. Mathématique et Informatique.

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