Quasi-rectilinear r-sets.

H. Desaulniers *
Département de mathématiques-informatique
Université du Québec à Trois-Rivières
N. F. Stewart †
Département d'informatique et de recherche opérationnelle
Université de Montréal

June 12, 1992

Abstract. One of the major difficulties in stating and proving theorems about the numerical stability of algorithms designed to effect operations, such as translation, rotation and regularized Boolean operations, on subsets of E^3, is that the actual subsets defined by set-representations are ill-defined. To correct this problem, we introduce the concept of *quasi-rectilinear r-set*, and describe, using a simple example, how they can be used to state and prove results of the kind described above.

1 Introduction

An *r-set* is a compact, regular, semi-analytic subset of Euclidean three-space E^3 [1, 2, 3]. The class of r-sets is a common choice for a modelling space in solid modelling. If an r-set S is actually equal to the underlying topological space $|K|$ of K, where K is a simplicial complex, then S will be called a *rectilinear* r-set [2]. In this paper r-sets in E^3 will often be referred to as "objects" or "solids".

There are several representational methods available for the representation of r-sets, or rectilinear r-sets, including Constructive Solid Geometry (CSG) [2] and Boundary Representations [4]. One of the difficulties with the latter approach, and it is this problem that is considered in this paper, is the possibility of inconsistency in the data provided, or inconsistency between the data and the underlying

*Full address: Département de mathématiques-informatique, Université du Québec à Trois-Rivières, Trois-Rivières, Québec, Canada G8Y 5S9. Electronic mail: helene@uqtr.uquebec.ca

†This research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada. Full address: Dép. IRO, Univ. de Montréal, C.P. 6128, Succ. A, Montréal, Québec, Canada H3C 3J7. Electronic mail: stewart@iro.umontreal.ca FAX (514)-343-5834

hypothesis about the class of objects represented. Such inconsistencies include, but are not limited to, those arising from *conversion* and *roundoff* error [5] due to the use of finite precision. In this paper we shall restrict our attention to the case of rectilinear r-sets, where there is (depending on the approach taken) either

- possible inconsistency between the actual geometric vertices and the hypothesis that faces are planar, or
- possible inconsistency between the actual geometric faces and the given symbolic information.

Similar (but more severe) difficulties occur in the case of boundary representations for r-sets with boundaries defined by points, curve segments, and trimmed surface patches [6]. A satisfactory solution to this problem is crucial, in particular, for the *numerical robustness* problem[1], which involves proving theorems about the subset of E^3 defined by the output of an algorithm using finite-precision floating-point arithmetic. It is futile to try to prove such theorems until the sets involved have been satisfactorily defined.

2 Boundary representations for rectilinear r-sets

A boundary representation for the rectilinear r-set S normally involves specification of a cell-decomposition [9], embedded in E^3, of the boundary ∂S of S. This specification usually involves vertices $\mathbf{v}^j \in E^3$, face-plane equations $\mathbf{n}^k \cdot \mathbf{x} = \xi_k$, and symbolic information Σ defining the (logical) relationships amongst the (logical) vertices, edges, and faces of ∂S. Typical examples are the "winged-edge" and related representations [4, 10, 11, 12]. However, because these data are not consistent amongst themselves, the actual subset S of E^3 is not well-defined. This is the inconsistency problem referred to in the Introduction.

One approach to the solution of this problem, in the case of rectilinear r-sets, is to restrict faces to be triangular [13]. However, boundary representations involving planar faces of general form are very widely used, and it is the sets defined by such representations that are the subject of this paper. A second approach is to view the geometric vertices $\mathbf{v}^j$ as approximate, and to suppose that S is defined by the face-plane equations $\mathbf{n}^k \cdot \mathbf{x} = \xi_k$, without reference to the vertices [14]. However, although it may be tolerable that the vertices $\mathbf{v}^j$ will not in general satisfy the face-plane equations, this approach suffers from other, more serious, defects. First of all, there is no guarantee that the face equations determine a rectilinear r-set, and secondly, even if they do, there is no guarantee [15, p. 21] that it corresponds to the symbolic information Σ. Furthermore, if these difficulties are eliminated by hypothesis, they reappear with crippling effect when even simple operations are applied to the objects [15, 16].

[1]Good overviews of this problem are given in [7, 8].

In contrast to these two approaches, we take the view that a rectilinear r-set is only a model for a real physical object, and that the uncertainty in the surfaces of the physical object will normally be large relative to binary conversion and other errors. We suppose also that the symbolic information Σ provided by the user is exact, and that the definition of the r-set $S \subseteq E^3$ must be rigorously consistent with this data. (Since the symbolic information is part of the description of the topological form of the object, it seems reasonable to assume that it has been specified exactly. Furthermore, since it is logical information, it is reliable, in the sense that it has not suffered conversion error on entry into the computer.) This leads us to the idea of a *quasi-rectilinear* r-set, the face F_k of the boundary of which is obtained by transfinite interpolation between the edges $[\mathbf{v}^j, \mathbf{v}^{j'}]$ of logical face k. Thus, the set S will be rigorously consistent with the symbolic information Σ, and with the geometric vertex $(\mathbf{v}^j)$ and edge $([\mathbf{v}^j, \mathbf{v}^{j'}])$ information[2]. The faces of S will not be exactly planar, but they will be nearly so: if the deviations from planarity, of the vertices in a single face, are small, then the set S will be close to a rectilinear r-set (and, in particular, normally much closer than the uncertainty in the surfaces of the real physical object being modelled).

The basic idea, of introducing perturbations of the given problem that are outside the class of nominal problems (in our case, those defined by rectilinear polyhedral sets), was apparently first suggested by Milenkovic [18, p. 19]. Note that it will never be necessary to actually construct the transfinite interpolants: it is only necessary to define them, in order to prove theorems about computational methods.

3 Quasi-rectilinear r-sets

We use a theorem of Whitney [19], McShane [20] and Aronsson [21] to construct the transfinite interpolant described above. The function $\psi_k(\mathbf{y})$ defining the geometric face F_k satisfies a Lipschitz condition with a constant that is proportional to the deviation from planarity of those geometric vertices $\mathbf{v}^j$ corresponding to face k. The domain F_k' of this function is defined by projecting the vertices $\mathbf{v}^j$ into the subspace orthogonal to the face normal $\mathbf{n}^k$. (In practice, we envisage situations where the $\mathbf{v}^j$ are nearly coplanar, so that there exists $\mathbf{t}^k$ such that $\|\mathbf{v}^j - pr(\mathbf{v}^j) - \mathbf{t}^k\|$ is small for each vertex $\mathbf{v}^j$ in the face.) See Figure 1.

If the vertices $\mathbf{v}^j$ are perturbations of the vertices of a rectilinear r-set, then the set S defined is close to the rectilinear r-set, as measured by the Hausdorff distance between the two sets, the Hausdorff distance between the boundaries of the two sets, and a pseudo-distance reflecting relative variation of the boundaries.

[2] This means that our interpretation is also a natural one in the sense that, even in the higher-dimensional case, specification of the vertices $\mathbf{v}^j$ is all that is required to specify a set [17].

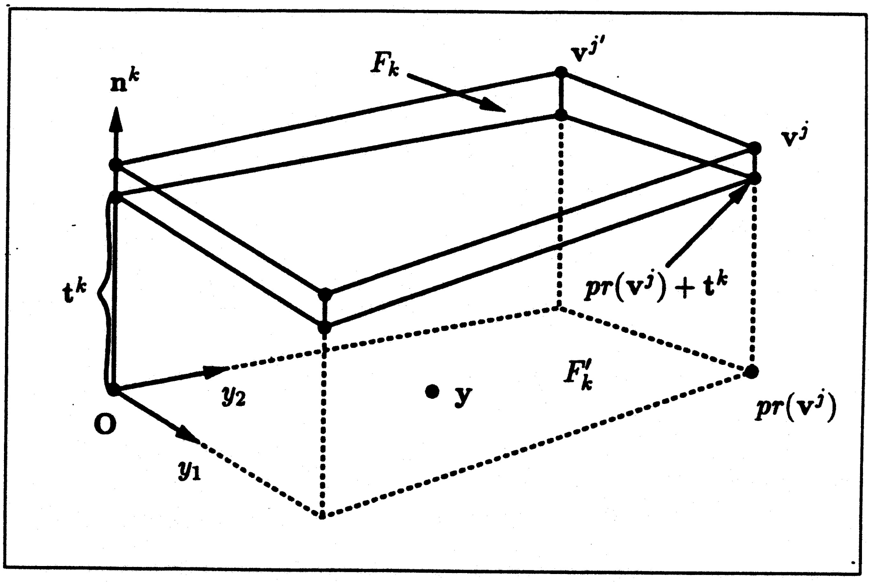

Figure 1: Construction of the face F_k

Furthermore, it can be proved that each point in S can be described [22, p. 450] in a neighbourhood of the point by a finite collection of analytic functions, and it follows that S is semi-analytic, and therefore an r-set.

4 Applications

Based on the concepts introduced above, it is possible to prove stability of certain algorithms implemented in ordinary finite precision arithmetic, provided the underlying problem is not ill-conditioned. It is also possible to use similar concepts to give a rigorous semantic interpretation of inconsistent data in the case of objects with curved faces. These applications are presented in detail elsewhere (in particular, see [23]). Here we only illustrate, using a simple example.

Consider the problem of translating the cube $C = \{\mathbf{x} : |x_i| \leq 0.1, i = 1,2,3\}$ by $\mathbf{t}$, where $t_i = 1, i = 1,2,3$. The cube is represented by the vertices $\mathbf{u}^j$ ($j = 1,...,8$) of the form $(\pm 0.1, \pm 0.1, \pm 0.1)$, normal vectors $\mathbf{n}^k$ ($k = 1,...,6$) of the form $(\pm 1,0,0)$, $(0,\pm 1,0)$, $(0,0,\pm 1)$, and the symbolic information linking the logical vertices, edges (twelve in number), and faces. Because of conversion error, the $\mathbf{u}^j$ are represented in the computer by $\mathbf{v}^j$, $\mathbf{v}^j \neq \mathbf{u}^j$, where $v_i^j = fl(u_i^j) = u_i^j(1+\delta_i^j)$, and $|\delta_i^j| < \epsilon \cong 10^{-7}, j = 1,...,8, i = 1,2,3$ [24, Ch. 3] [25, p. 198]. The process of transfinite interpolation described above defines a quasi-rectilinear r-set C' such that $d(C, C')$ and $d(\partial C, \partial C')$ are on the order of ϵ, where d denotes the Hausdorff distance.

Let distance between two non-empty compact sets A and B in E^3 be measured by the maximum of the Hausdorff distance between the sets, $d(A, B)$, and the Hausdorff distance between the boundaries of the sets, $d(\partial A, \partial B)$, with the additional condition that the distance is infinite if there does not exist a homeomorphism of E^3 onto E^3 that carries A onto B [26, 27, 28]. (That is, if two objects are to be close, they must, in a very strict sense, have the same topological form). With this definition, the problem of computing set translation is well conditioned. Furthermore, effecting the translation by calculating $fl(v_k^j + t_k^j)$ defines a (computed) translated set T' such that the distance, as defined above, from T' to $T \equiv C + \mathbf{t}$, is also on the order of ϵ. Consequently, the simple method for translating a set is numerically stable. This result is intuitively obvious; however, it is not possible to state (much less prove) rigorously even straightforward theorems such as this, until we have defined the actual subsets of E^3 being manipulated by the algorithms involved. This has been done by the introduction of quasi-rectilinear r-sets.

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