Segment Visibility Graphs: Several Results

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Abstract

We explore two primary types of segment visibility graphs: those whose nodes are the segment endpoints, and those whose nodes are the segments themselves. Several subclasses of these graphs can be identified by specializing the class of segments considered, or restricting the type of visibility permitted. We obtain several new results on the structure of these visibility graphs.

1 Introduction

We call the two primary graphs mentioned in the abstract "endpoint visibility graphs" and "segment visibility graphs."

2 Endpoint Graphs

An *endpoint visibility graph* G of a set S of closed, disjoint line segments has a node for each segment endpoint, and an arc between two nodes x and y if $[x,y] \cap S = [x,y]$ or $\{x,y\}$, where $[x,y]$ is the segment between x and y. We say that the two endpoints x and y are *visible* to each other, or that they *see* each other. Note that G contains an arc corresponding to each segment in S.

Here the focus of our attention has been the outstanding conjecture[1] that the endpoint visibility graph for a set of noncollinear disjoint segments has a "simple" Hamiltonian cycle, one that does not self-cross when its nodes are embedded at their corresponding endpoints. Mirzaian first proved this for what we call *hulled* segments:[2] segments each of which touches the convex hull of the segments [Mir92].[3] We offer four new results on this conjecture, each for a special class of segments.

1. *Hulled Segments*

   We offer a different proof of Mirzaian's result on hulled segments: that the endpoint visibility graph always includes a *circumscribing Hamiltonian circuit* or *circumscribing polygon*, i.e., a circuit that is simple and circumscribes the segments. Our proof does not assume noncollinear endpoints as does his, so establishes the result for a somewhat wider class of segments.[4]

   We prove a particular naive algorithm always finds the circuit: "wrap" the segments in order around the hull in a sawtooth fashion, and adjust the path in the obvious way when it touches itself. See Fig. 1. The algorithm is simple but the proof of correctness is not.

2. *Independent Segments*

   We define a set of segments to be *independent* if for each s in the set, the line containing s does not meet any other segment in the set. For this class we prove the endpoint visibility graph always includes a

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[1] This conjecture has been formulated by several researchers independently: Mirzaian [Mir92], Toussaint [Tou92], and (later) in [OR91].

[2] This class of segments was introduced by Toussaint in 1985 [Tou92].

[3] If all segments are collinear, they are hulled but have no Hamiltonian cycle with our definition of visibility.

[4] Although this assumption is sometimes critical for visibility graph structure, it seems likely that the assumption of nondegeneracy in [Mir92] is inessential.

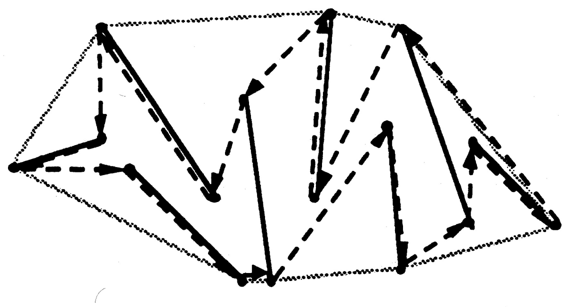

Figure 1: Hamiltonian cycle found by hulled segments algorithm [OR91].

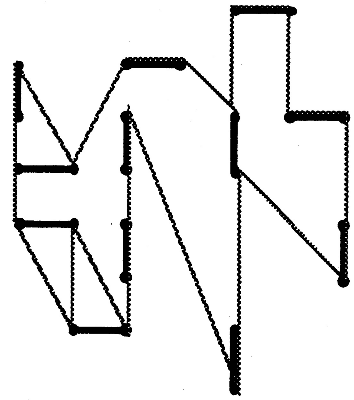

Figure 2: Simple Hamiltonian circuit through unit lattice segments.

circumscribing Hamiltonian circuit. This is a relatively easy result.

3. *q-Orderable Segments*

   This somewhat unnatural class demands that there exists an ordering of the segments $s_1, s_2, \ldots$ such that s_i can form a quadrilateral q with s_{i-1} using visibility edges, such that no quadrilateral edges determined by the segments up to s_{i-1} cross q. Here we can prove the endpoint visibility graph always includes a simple Hamiltonian circuit. This result is not difficult, but points toward more natural classes of segments (we call one "shellable") for which we could not establish the conjecture.

4. *Unit Lattice Segments*

   For any set of disjoint segments with endpoints on the integer lattice, and each of unit length (so all segments are vertical or horizontal), we prove that the endpoint visibility graph always includes a simple Hamiltonian circuit. See Fig. 2. Despite the highly constrained nature of this class of segments, our proof of this result is nontrivial.

3 Segment Graphs

The ***segment visibility graph*** for a set of disjoint segments associates a node for each segment and an arc between segments a and b if some point of a can see some point of b. Here we examined three classes, dependent upon visibility restrictions: what we call H-visibility graphs, HV-representations, and unrestricted segment graphs.

3.1 H-visibility graphs

This is the most-studied segment visibility graph: visibility is restricted to a single direction, which we take to be horizontal. So there is an arc between segments a and b iff there is a horizontal segment between a and b that does not touch any other segment. One reason this class of graphs has been scrutinized is their obvious relevance to sweepline algorithms.

For these graphs, as with many visibility graphs, the structure is very much dependent upon whether or not three endpoints of the segments are collinear. For vertical segments with no collinearities, Luccio *et al.* obtained a characterization in terms of a certain multigraph [LMW87]. For vertical segments with collinearities permitted, Tamassia and Tollis [TT86] and Andreae [And89] obtained complex characterizations, but Andreae proved that the recognition problem is NP-complete and so no "good" characterization is likely.

Here we offer two new results.

1. *Slanted ≡ Vertical*

   We prove that the H-visibility graph for every set of segments whose endpoints have distinct vertical coordinates, is realized by a set of vertical segments. Thus slanted (nonvertical) segments yield the same class of graphs as vertical segments under the noncollinearity assumption.

2. *Characterization*

   The class of graphs realized as H-visibility graphs of segments whose endpoints have distinct vertical coordinates, is precisely planar graphs that have an embedding such that, for every interior k-face in the graph, the induced subgraph of the k-face has exactly $2k-3$ edges.

Both of these results are nearly implicit in [LMW87].

3.2 HV-Representations

As a median between H-visibility graphs and unrestricted segment visibility graphs, we examined two directions of visibility, which we take (without loss of generality) to be horizontal and vertical. We obtained two results.

1. *G mating with G*

   If a graph G can be realized as both the H-visibility graph of a set of segments, and simultaneously as the vertical V-visibility graph of the same set of segments, then we say it has an *HV-visibility representation*. If collinearities are forbidden, then we prove that every graph that has an H-visibility representation also has an HV-visibility representation. If, however, collinearities are permitted, we do not know whether existence of an H-visibility representation implies existence of an HV-visibility representation.

2. *Pairs of graphs*

   Generalizing to pairs of unequal graphs, we say that (G, G') have an HV-visibility representation if there is a set of segments whose H-visibility graph is G and whose V-visibility graph is G'. Here we only know that if collinearities are permitted, there exist pairs of graphs, each representable with one direction of visibility, which do not have an HV-visibility representation.

3.3 Segment Visibility Graphs

For unrestricted visibility, we obtained two results and formulated one conjecture.

1. *Connection to Polygon Visibility Graphs*

   We noticed a curious connection to vertex visibility graphs of polygons, seemingly rather distantly related. Nevertheless ElGindy's result [O'R87] that the class of polygon visibility graphs includes all mops[5] can be reinterpreted to say that the class of segment visibility graphs realized by vertical segments includes all mops.

   Similarly, the recent characterization of staircase visibility graphs [AEK91] can be interpreted as a characterization of segment visibility graphs for a subclass of vertical segments.

2. *K_4's*

   We can make one observation about the general structure of segment visibility graphs. Let S be a set of $n > 1$ segments such that no three endpoints are collinear, and let G be its segment visibility graph. Add an infinite vertical segment to the left and one to the right of S, and call this augmented set S' and its segment visibility graph G'. Then we prove that, for every vertex $a \in G$, there exist three other vertices $b, c, d \in G'$ such that the induced subgraph of a, b, c, and d in G' is K_4. The proof is not difficult, but at least shows a certain density of edges incompatible with planarity.

3. *Perfect Matching Conjecture*

   Although there are easy examples of segments whose graph contains no Hamiltonian path, we have no counterexample to this hypothesis: the segment visibility graph for every set of segments has a perfect matching.[6] We have proven it for several classes of segments, e.g., independent segments.

References

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[5] A *mop* is a maximal outerplanar graph, i.e., a polygon triangulation.

[6] A *perfect matching* is a set of arcs that matches every node exactly once. We extend the definition to graphs with an odd number of nodes by allowing one unmatched node.

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