Sorting Does Not Always Help in Computational Geometry*

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Abstract

We show that $\Omega(nlogn)$ remains a lower bound for the computation of Closest Pair, Euclidean Minimum Spanning Tree, Voronoi Diagram, Closest Pair Between Two Sets and the Largest Empty Circle, even if the points are sorted lexicographically on (X,Y). On the other hand, it is already known that sorting does help computing many other geometric problems: Convex Hull, Triangulation of n points, Diameter of n points and the Maxima of n points. Our results show that the former class of problems is intrinsically more difficult than the latter, although they have the same lower bounds in the absence of sortedness.

1 Introduction

Proving lower bounds is a central part of theory of algorithms and computational complexity. In this paper, we consider the following problems : given additional information in the input to some problems whose lower bounds are already known, will the lower bounds still hold?

Seidel [7] was probably the first one who formally considered such problems. For example, he showed that given n points in 3-dimensions, even if the sorted ordering of the n points along all three axes are known, computing the convex hull of the n points in 3-dimensions still has a lower bound of $\Omega(nlogn)$.

These kinds of problems are meaningful in theory as well as in practice. For example, there are some $\Theta(nlogn)$ time algorithms to construct the Voronoi Diagram of n points in the plane([3],[4] and [6]). But can we reduce the upper bound down to $o(nlogn)$ in the presence of some additional information? As of this writing, only the case when the points form a convex polygon has been solved in linear time by Aggarwal, et al. in [1]. We will show in this paper that the lower bound for constructing the Voronoi Diagram for a set of n points remains $\Omega(nlogn)$ even if the given points are lexicographically sorted along the coordinate axes.

Throughout this paper, when we mention points, we mean that they are planar points. The *lex-ordered* version of a problem A(whose input is a set of planar points) is the problem A when the input points are given in lexicographical order along the coordinate axes(i.e, (X,Y)). In this paper, we will consider the lex-ordered version of the following problems: Closest Pair, Euclidean

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Minimum Spanning Tree(EMST for short), Voronoi Diagram, Closest Pair Between Two Sets and the Largest Empty Circle. We will show that the lower bound of lex-ordered version of the above problems remains $\Omega(nlogn)$. The basic techniques we use are reduction from the *Integer Minimum Gap*, reduction from the *Integer Set Disjointness* and reduction from the *Integer Maximum Gap*. These three problems have the $\Omega(nlogn)$ lower bound in the algebraic computation tree model, as shown in [8].

2 Lower bound for the lex-ordered Closest Pair, Voronoi Diagram and EMST

In this section, we will first give the following result:

Theorem 1: *The lex-ordered Closest Pair has a lower bound of $\Omega(nlogn)$ in the algebraic computation tree model.*

Proof: Suppose we are given an arbitrary list of n distinct integers $Y = \{y_1, y_2, ..., y_n\}$. We construct the corresponding list of points $P=\{p_i | p_i = (i, ny_i), 1 \leq i \leq n\}$. It is easy to see that the points in P are already sorted lexicographically along (X,Y). We can show that (p_i, p_j) is the closest pair of P if and only if the vertical distance $|y(p_i) - y(p_j)|$ is minimum possible, in other words, their vertical distance is the Minimum Gap of their y-coordinates. Therefore, the Minimum Gap problem of $y_i (1 \leq i \leq n)$ can be transformed in linear time to the Closest Pair problem for $p_i (1 \leq i \leq n)$. □

It is also known that the problem of computing the Closest Pair of an n-vertex simple polygon(i.e, the order in which the n points can form a simple polygon is known and the edges of the simple polygon are not considered as barriers) has a lower bound of $\Omega(nlogn)$ in the algebraic computation tree model [8]. Here we give a stronger result than that:

Corollary 2:*The lower bound for computing the Closest Pair of n points remains $\Omega(nlogn)$ in the algebraic computation tree model even if the order in which the n vertices form a monotone polygon is known.*

Proof: Since the points are sorted along the X-axis, we can construct an X-monotone polygon in linear time with these points as vertices. Any $o(nlogn)$ time algorithm to find the Closest Pair of the X-monotone polygon will imply an $o(nlogn)$ time algorithm for the closest pair of the n points. □

Now we can simply have the results on the lex-ordered Voronoi Diagram and EMST problems:

Theorem 3: *The lex-ordered Voronoi Diagram problem has a lower bound of $\Omega(nlogn)$ in the algebraic computation tree model.*

Proof : The proof is by a reduction from the lex-ordered Closest Pair problem. □

Since the Closest Pair problem can be transformed to the Euclidean Minimum Spanning Tree problem in linear time, we give the following results without proof.

Theorem 4: *The lex-ordered EMST problem has a lower bound of $\Omega(nlogn)$ in the algebraic computation tree model.*

In [1], Aggarwal et al. gave a linear time algorithm to construct the Voronoi Diagram of a convex polygon. Here we want to ask the following question: Is $\Omega(nlogn)$ a lower bound for computing the Voronoi Diagram(or Delaunay Triangulation) of the vertices of an n-vertex simple polygon(the edges of the simple polygon are not considered as barriers)? It turns out that the lower bound will remain $\Omega(nlogn)$ even if the n points form a monotone polygon.

Theorem 5 : *The lower bound for computing Voronoi Diagram of n points remains $\Omega(nlogn)$ in the algebraic computation tree model even if an order in which the n vertices form a monotone polygon is known.*

Proof: By a reduction from the Closest Pair problem of a monotone polygon(i.e., the problem we discussed in Corollary 2). □

3 Lower bound for computing the lex-ordered Closest Pair Between Two Sets

The problem of computing the Closest Pair between two sets of points has been proved by D.T. Lee and F. Preparata to have a lower bound of $\Omega(nlogn)$ even if the two sets are known to be linearly separable [5]. Here we will give the following lower bound result:

Theorem 6: *The lex-ordered Closest Pair Between Two Sets problem has a lower bound of $\Omega(nlogn)$ in the algebraic computation tree model.*

Proof: We use a reduction from the *Integer Set Disjointness* problem [8]. Suppose we are given two sets of integers, $A = \{x_1, ..., x_n\}$, $B = \{y_1, ..., y_n\}$. We want to determine whether $A \cap B = \emptyset$ or not. We construct two sets of points as follows, $\alpha = \{(2i-1, n^2x_i)|1 \leq i \leq n\}$ and $\beta = \{(2i, n^2y_i)|1 \leq i \leq n\}$. We can see that $A \cap B \neq \emptyset$ if and only if the length of the closest pair between α and β is smaller than or equal to $2n-1$. That is, the Integer Set Disjointness problem is linear time transformable to the Closest Pair between two sets of lex-ordered points. □

4 Lower bound for computing the lex-ordered Largest Empty Circle

The largest empty circle problem is defined as follows: given n points in the plane, find the largest circle whose center is in the convex hull of the points and which does not contain any of the points in its interior. This problem has been known to have a lower bound of $\Omega(nlogn)$ [5]. We give the following result:

Theorem 8: *The lex-ordered Largest Empty Circle problem has a lower bound of $\Omega(nlogn)$ in the algebraic computation tree model.*

Proof: The proof is by a reduction from the *Integer Maximum Gap* problem([8]). Suppose we are given a set of n integers, $A = \{x_1, ..., x_n\}$. We want to compute the Maximum Gap of A. Construct a set of n points as follows: $P = \{(i, n^2x_i)|1 \leq i \leq n\}$. We can show that, (1) the Largest Empty Circle cannot pass through three points in P(otherwise, the center of the circle will be out of the convex hull of P); (2) the Largest Empty Circle must pass through two points of P that are consecutive along the Y-axis; and (3) the Largest Empty Circle passes through two points (k, n^2x_k) and (l, n^2x_l) of P if and only if the difference between the two integers x_k and x_l is the Maximum Gap of A. □

5 Concluding remarks

First, we give two directions for these kinds of problems for further research :

1. If the sorted ordering along both the x- and the y-coordinates of the n points are given, will the lower bounds for computing the above geometric problems still be $\Omega(nlogn)$?

2. Compute the lower bound(or upper bound) for some geometric problems when some additional information other than sortedness is given. These kinds of problems have been mentioned in the literature, but little results is known. For example, the following problem was listed as an open problem in [2]: if the Euclidean Minimum Spanning Tree of the n points is given, will the lower bound for constructing the Voronoi Diagram of the n points still be $\Omega(nlogn)$?

Finally, the following table summarizes the state of knowledge about whether sortedness helps computing certain geometric problems.

Problem	*lower bound*	*Does sorting help?*	*Results after sorting*
1. Convex Hull	$\Omega(nlogn)$	Yes	$O(n)$
2. Triangulation of n points	$\Omega(nlogn)$	Yes	$O(n)$
3. Diameter of n points	$\Omega(nlogn)$	Yes	$O(n)$
4. Maxima of n points	$\Omega(nlogn)$	Yes	$O(n)$
5. Closest Pair	$\Omega(nlogn)$	No	$\Omega(nlogn)$
6. Voronoi Diagram	$\Omega(nlogn)$	No	$\Omega(nlogn)$
7. Largest Empty Circle	$\Omega(nlogn)$	No	$\Omega(nlogn)$
8. Closest Pair of Two sets	$\Omega(nlogn)$	No	$\Omega(nlogn)$
9. EMST	$\Omega(nlogn)$	No	$\Omega(nlogn)$
10.Closest Pair of Two linearly separable sets	$\Omega(nlogn)$	Unknown	$O(nlogn)$
11.Collinearity problem [5]	$\Omega(nlogn)$	Unknown	$O(n^2)$
12.Discrete 1-center problem [5]	$\Omega(nlogn)$	Unknown	$O(nlogn)$

References

[1] A. Aggarwal, L.J. Guibas, J. Saxe, and P.W. Shor. A linear-time algorithm for computing the voronoi diagram of a convex polygon. *Discrete and Computational Geometry*, 4(6):591–604, 1989.

[2] A. Aggarwal and J. Wein. *Computational Geometry : Lecture Notes for 18.409*. Lab. for Computer Science, MIT, Spring, 1988.

[3] S.J. Fortune. A sweepline algorithm for voronoi diagrams. *Algorithmica*, 2(2):153–174, 1987.

[4] L.J. Guibas and J. Stolfi. Primitives for the manipulation of general subdivisions and the computation of voronoi diagrams. *ACM Trans. Graphics.*, 4(2):74–123, 1985.

[5] D.T. Lee and F.P. Preparata. Computational geometry — a survey. *IEEE Trans. Comp.*, C-33(12):1072–1101, 1984.

[6] F.P. Preparata and M.I. Shamos. *Computational Geometry: An Introduction*. Springer-Verlag, 1985.

[7] R. Seidel. A method for proving lower bounds for certain geometric problems. In G.T. Toussaint, editor, *Computational Geometry*, pages 319–334, 1985.

[8] A.C.C. Yao. Lower bounds for algebraic computation trees with integer inputs. In *Proc. 30th IEEE FOCS*, pages 308–313, 1989.