On the Zone of a Co-dimension p Surface in a Hyperplane Arrangement*

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Abstract

A recent work of Aronov and Sharir gives almost tight bound on the complexity of all cells in an arrangement of hyperplanes that are intersected by an algebraic surface of co-dimension 1. We extend their result to the case of a surface of co-dimension p. The upper bound is $O(n^{d-\lceil p/2\rceil}\log^{\epsilon_p} n)$. Where ϵ_p is 0 for p even, 1 for p odd. The lower bound is $\Omega(n^{d-\lceil p/2\rceil})$. The upper bound is tight for even p, and almost tight for odd p.

1 Introduction

A recent result of Aronov and Sharir on the zone of a surface in an arrangement of hyperplanes in [AS91] is the following: any algebraic surface of bounded degree and of co-dimension 1 intersects cells in an arrangement of hyperplanes whose total complexity is $O(n^{d-1}\log n)$. In this paper we generalize the result of [AS91] to surfaces of co-dimension p, $0 \leq p \leq d$. We aim to achieve a complexity that is roughly $O(n^{d-\lceil p/2\rceil})$ since this is a lower bound for this problem.

2 Lower bound construction

The lower bound is meaningful for $d \geq p$. We prove the lower bound by induction on d. For $d = p$ a co-dimension p surface is a point. The cell containing the point can have complexity $\Theta(n^{\lfloor p/2\rfloor}) = \Theta(n^{p-\lceil p/2\rceil})$ by the upper bound theorem for simple polytopes [Ede87]. This proves the bound for $d = p$. Assuming there is a construction for dimension $(d-1)$ attaining the bound, we can extend every hyperplane and the surface orthogonally in the d-th dimension. Moreover we introduce a linear number of hyperplanes orthogonal to the x_d-axis.

Every cell in the original $(d-1)$-dimensional is replicated n times. And every cell intersected by the surface on R^{d-1} generates cells cut by the surface in dimension R^d. Therefore we obtain the bound $\Omega(n^{d-1-\lceil p/2\rceil}n) = \Omega(n^{d-\lceil p/2\rceil})$.

3 A geometric lemma

Lemma 1 *Given a simple hyperplane arrangement $\mathcal{A}(H)$, a face f of $\mathcal{A}(H)$, and a point $v \notin f$, let u be the point in f closest to v, then there is a cell C incident to f such that u is the point of C closest to v.*

Proof.

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1. If f is a vertex, then $f = u$. Let H_f be the set of d hyperplanes in R^d meeting in u. We show that there is one cell in the cell complex $\mathcal{A}(H_f)$ for which u is its closet point to v. Note that all cells in $\mathcal{A}(H_f)$ are cones with one vertex, therefore we talk of a *cone complex*.

   We define $h^+_{\vec{b},a} = \{x | (x-a) \cdot \vec{b} \geq 0\}$, in words, the halfspace supported by the hyperplane through a orthogonal to $\vec{b}$ and containing the point $a + \vec{b}$. For a cell C in $\mathcal{A}(H_f)$, the *complementary cone* C' is :

   $$C' = \bigcap_{q \in C, q \neq u} h^+_{u-q,u}$$

   Intuitively, C' is formed by taking halfspaces through u whose opposite normal vector is inside C. We prove the following:

   (a) If $v \in C'$ then u is closest to v than to any other point in C.

   (b) The set $\mathcal{P}'$ of complementary cones of cones in a cone complex $\mathcal{P}$ is a cell complex covering all R^d.

   Consider a point $u' \in C$ with $u' \neq u$. Since $v \in C'$ the hyperplane $h_{u,u-u'}$ separates v from C. The angle $\widehat{u'uv}$ is greater than or equal to $\pi/2$. This implies that the segment $u'v$ is longer than the segment $\bar{u}v$.

   From Lemma 3.1 (2) in [Cla87], a complementary cone is generated by intersecting halfspaces corresponding to the directions of edges of C (defining $extrC$ the set of vectors corresponding to edges of C, $C' = \bigcap_{b \in extrC} h^+_{b,u}$).

   A set of d hyperplanes generates d 1-flats: each 1-flat generates one hyperplane and 2 halfspaces. Under this correspondence between edges and halfplanes, the set of edges of $\mathcal{P}$ induce a cell complex $\mathcal{P}''$ with 2^d cells. Since the correspondence between cones in $\mathcal{P}$ and $\mathcal{P}'$ is 1-1 we have that the set $\mathcal{P}'$ of complementary cones is exactly the set $\mathcal{P}''$ and, by construction, $\mathcal{P}'$ covers the whole space.

2. Suppose $u \in int(f)$. Let f be a face of dimension k (i.e. it is contained in the intersection of a set (H_f) of $d-k$ hyperplanes). If $u \in int(f)$ then u is the point of $aff(f)$ closest to v. The set of hyperplanes H_f divide the space into 2^{d-k} regions.

   A set P is polyhedral if it is the intersection of a finite set of halfspaces. In [Gru67] it is proved that any polyhedral set admits a representation $P = L^{\perp} + (L \cap P)$, where L is a linear subspace, $L^{\perp}$ is its orthogonal complement, $+$ is the pointwise sum of point-sets, and $L \cap P$ is a polyhedral set whose faces have at least one vertex.

   Similarly, $(d-k)$ hyperplanes generate a cell complex $\mathcal{P}$ of 2^{d-k} cells sharing a common k-flat. $\mathcal{P}$ admits a represation $\mathcal{P} = L^{\perp} + (L \cap \mathcal{P})$ where $L^{\perp}$ is the common k-flat and L is an orthogonal $(d-k)$-flat. We can choose L such that $u \in L$ and $uv \subset L$. Moreover, $L \cap \mathcal{P}$ is a cone complex of cones sharing a common point, which is u.

   Using part 1. of this proof we find a cell C in $\mathcal{P} \cap L$ for which u is the closest point to v. Let q be any point in the cell $C + L^{\perp}$. Clearly, there is a point $q' \in C$ such that $q \in q' + L^{\perp}$. Using Pitagora's theorem $|vq'| \leq |vq|$, therefore $|vu| \leq |vq|$.

3. Suppose $u \notin int(f)$ and f has dimension $k > 0$.

   If $u \notin int(f)$ then there is a face g, which is facet of f, such that $u \in g$. Also, u is the closest point in g to v. By induction on k we have a cell C_g which has u as its closest point to v. Cell C_g is adjacent to g and to the hyperplanes in H_g. Cell C_g is adjacent also to $H_f \subset H_g$ and therefore it is adjacent to f.

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4 Proof of the Zone theorem for co-dimension p surfaces

For a d-polyhedron P, let $f_k(P)$ be the number of k-faces of P (i.e. faces of dimension k). Let $Z_\sigma(H)$ be the set of cells in $\mathcal{A}(H)$ whose *relative interior* has a non empty intersection with $\sigma \subseteq R^d$.

We denote $z_k(\sigma, H)$ the $\sum_{C \in Z_\sigma(H)} f_k(closure(C))$. We set $n > 0, d > 0, 0 \leq k \leq d$, and by $z_k(n,d)$ we denote the maximum of $z_k(\sigma, H)$ over all σ algebraic surfaces of bounded degree δ and co-dimension p, and all sets of n hyperplanes. First we notice that a standard perturbation argument proves that the maximum of $z_k(n,d)$ is attained when the hyperplanes in H are in general position and σ is in general position with respect to H [Gru67]. Under the general position assumption, a k-face f in $\mathcal{A}(H)$ lies in exactly $(d-k)$ hyperplanes and is part of the boundary of 2^{d-k} cells of $\mathcal{A}(H)$. More than one of those cell can lie on $Z_\sigma(H)$, thus the contribution of f to $z_k(\sigma, H)$ can be more than one.

We define a *k-border* as a pair (f, C), where f is a k-face and C a cell having f on its boundary. Thus $z_k(\sigma, H)$ counts all k-borders in $Z_\sigma(H)$ once. More generally, a (k,i)-border, $0 \leq k \leq i \leq d$ is a pair of faces (f, g) of dimensions k and i respectively, such that $f \subset closure(g)$. Note that k-borders are (k,d)-borders.

We call an i-face *popular* if all the 2^{d-i} incident cells are in the zone $Z_\sigma(H)$. A (k,i)-border (f,g) is popular if g is a popular i-face.

Definition 1 *$\tau_k^i(X, H)$ is the number of popular (k,i)-borders in the zone of $X \subseteq R^d$ in the arrangement of H.*

Note that $z_k(\sigma, H) = \tau_k^d(\sigma, H)$. So by estimating $\tau_k^d(\sigma, H)$ for each k, $0 \leq k \leq d$ we find the total complexity of the zone. We obtain such bounds inductively estimating τ_k^i, for all $0 \leq k \leq i \leq d$.

Lemma 2 *1. For any subset $X \subset R^d$ and $0 \leq k \leq d$*

$$\tau_k^k(X, H) \leq \binom{d}{k} \tau_d^d(X, H)$$

2. For an algebraic surface σ of co-dimension p and bounded degree,

$$\tau_k^k(\sigma, H) = O(n^{d-p})$$

Proof.

1. As noticed in [AS91] it suffices to associate any popular k-face with a popular cell and argue that each cell cannot be charged too many times. Pick up a point u in the interior of a cell of the arrangement. For any convex set C, not containing u, there exists one and only one point $v \in C$ such that $|uv| = \min_{q \in C} dist(u, q)$, where $dist$ is the standard euclidean distance

in R^d. Take now a popular k-face f and find its point v at minimum distance from u. We konw from Lemma (1) that there exists one cell in $Z_\sigma(H)$ incident to f, such that v is its point at minimum distance from u. We associate f with this cell. Now each popular k-face is associated with a cell with which shares its closest point to u. No cell in the zone can be charged more than $\binom{d}{k}$ times (that is when v is a vertex), if v is not a vertex there are even fewer possible k-flats incident on v.

2. It is enough to show that $\tau_d^d(\sigma, H) = O(n^{d-p})$ (i.e. σ meets $O(n^{d-p})$ cells of $\mathcal{A}(H)$). We prove this by induction on d. For $d = p$ a co-dimension p surface is a set of points whose number depend on the degree δ. Therefore, from the definition of $Z_\sigma(H)$, the number of cells is $O(1)$. Otherwise suppose inductively that, on each hyperplane $h \in H$, $\sigma \cap h$ meets $O(n^{d-1-p})$ cells in the arrangement induced on h by H. The surface σ can intersect the cell C in 2 cases: when a component of σ is fully contained in C and when σ crosses the boundary of C. The former case can happen only a constant times , the latter case is bounded by $O(nn^{d-1-p}) = O(n^{d-p})$. The number of cells in $Z_\sigma(H)$ is $O(n^{d-p})$.

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The proof of lemma 2 (1) is a key change with respect to [AS91] since we do not have any extra correction term polynomial in n.

Now we proceed by induction on i to derive a recurrence for $\tau_k^i(\sigma, H)$, for $0 \leq k < i$. Fix an hyperplane $h \in H$ and consider a popular (k,i)-border (f_0, g_0) in $Z_\sigma(H)$, with $f_0 \notin h$. When we remove h, g_0 becomes a possibly bigger i-face g which is also popular, moreover f_0 is a part of some k-face in $closure(g)$. So let (f,g) be a popular (k,i)-border in $Z_\sigma(H - \{h\})$. We consider what happens when h is reintroduced. Let C_l, for $l \in [1, \ldots, 2^{d-i}]$ be the cells of $Z_\sigma(H - \{h\})$ incident to g. The following cases may occur:

1. $h \cap g = \emptyset$, in this case g may or may not be popular in $Z_\sigma(H)$. In the first case (f,g) contributes one (k,i) border to the zone. In the second case (f,g) is not counted any more.

2. $h \cap g \neq \emptyset$ and $h \cap f = \emptyset$, again (f,g) contributes at most 1, namely (f, g^+) where g^+ is the portion of g on the same side of h as f.

3. $h \cap g \neq \emptyset$ and $h \cap f \neq \emptyset$, in this case we get 2 (k,i)-borders: $(f \cap h^+, g \cap h^+)$ and $(f \cap h^-, g \cap h^-)$. Only if both of them are popular our count will increase, i.e., let $C_l^+ = C_l \cap h^+$ and $C_l^- = C_l \cap h^-$, if σ meets all these 2^{d-i+1} cells . Notice that all these cells are adjacent to $g \cap h$ which is an $(i-1)$-face in $\mathcal{A}(H)$. The count will increase then if $g \cap h$ is a popular $(i-1)$-face in $Z_\sigma(H)$ and $(f \cap h, g \cap h)$ is a popular $(k-1, i-1)$-border in $Z_\sigma(H)$.

To summarize: the number of popular (k,i)-borders not contained in h is bounded by $\tau_k^i(\sigma, H - \{h\}) + \rho_h$, where ρ_h is the number of popular $(k-1, i-1)$-borders (f', g') with $g' \subset h$. Summing over all $h \in H$ we have that every popular (k,i)-border is counted exactly $n - d + k$ times. We obtain an equation:

$$(n - d + k)\tau_k^i(\sigma, H) \leq \sum_{h \in H} \tau_k^i(\sigma, H - \{h\}) + (d - i + 1)\tau_{k-1}^{i-1}(\sigma, H)$$

Where the factor $(d-(i-1))$ comes from the fact that we charge a popular $(i-1)$-face $d-(i-1)$ times, i.e any time h is an hyperplane containing it. Maximizing over all arrangements we get:

$$\tau_k^k(n,d) = O(n^{d-p}) \tag{1}$$

$$\tau_k^i(n,d) \le \frac{n}{n-d+k}\tau(n-1,d) + \frac{d-i+1}{n-d+k}\tau_{k-1}^{i-1}(n,d) \tag{2}$$

Note that this is an induction in n, i and k, not in d. We assume that $n \ge d-k$ and define $\tau_k^i(n,d) = \binom{n}{d-k}\psi_k^i(n,d)$. Equations (1) and (2) become:

$$\psi_k^k(n,d) = O(n^{k-p}) \tag{3}$$

$$\psi_k^i(n,d) \le \psi_k^i(n-1,d) + \frac{d-i+1}{d-k+1}\psi_{k-1}^{i-1}(n,d), \qquad 1 \le k < i \le d \tag{4}$$

Equation (4) can be rewritten also in the following form:

$$\psi_k^i(n,d) \le \psi_k^i(n-1,d) + \frac{c}{n}\frac{\tau_{k-1}^{i-1}(n,d)}{n^{d-k}} \tag{5}$$

The base case $i = 0, k = 0$ is dealt with by lemma 2. Similarly the case $i = 1, k = 0, 1$ are dealt with by lemma 2 and by the observation that $\tau_0^i(n,d) \le 2\tau_1^i(n,d)$, that is, vertices can be charged to the edges.

Let us suppose now that $i < p$. In this case the number of popular i-faces is $\tau_i^i(n,d) = O(n^{d-p})$. The maximum complexity of an i-polytope is $O(n^{\lfloor i/2 \rfloor}) \le O(n^{\lfloor p/2 \rfloor})$. Therefore for every $k \le i$ $\tau_k^i(n,d) = O(n^{d-\lceil p/2 \rceil})$.

Now we solve the recurrence for $i = p$ and $k = \lceil p/2 \rceil$. When we have this bound, it can be extended to every k for $0 \le k \le p$, using the fact, consequences of the Dehn-Sommerville relations, that the number of $\lceil p/2 \rceil$-faces bounds the number of faces of any dimension [Ede87,AMS91].

Consider equation (5). We estimate up to a constant factor $\tau_{k-1}^{i-1}(n,d) \le n^{\lfloor (i-1)/2 \rfloor}\tau_i^i(n,d) \le n^{\lfloor (p-1)/2 \rfloor}n^{d-p}$. So the fraction in equation (5) becomes $n^{\lfloor (p-1)/2 \rfloor - \lfloor p/2 \rfloor}$, which is constant for p odd, and n^{-1} for p even. Therefore the additive term in the equation becomes $1/n$ or $1/n^2$. We obtain $\psi_k^i(n,d) = \log n$ or $\psi_k^i(n,d) = O(1)$. Easily follows that $\tau_k^i(n,d) = O(n^{d-k}\psi_k^i(n,d)) = O(n^{d-\lceil p/2 \rceil})\psi_k^i(n,d)$.

For $i > p$ we solve the recurrence (5) assuming $k \ge \lceil p/2 \rceil$. Assume that the bound holds inductively, and p is even, so $\psi_{k-1}^{i-1}(n,d) = O(n^{k-1-\lceil p/2 \rceil})$. Inserting the bound in equation (4) we obtain $\psi_k^i(n,d) = O(n^{k-\lceil p/2 \rceil})$ which gives the final bound $\tau_k^i(n,d) = O(n^{d-\lceil p/2 \rceil})$. For $k < \lceil p/2 \rceil$ we have $k < \lceil p/2 \rceil < \lceil i/2 \rceil$. From the Dehn-Sommerville equations we know that for all k's $\tau_k^i(n,d) = O(\tau_{\lceil i/2 \rceil}^i(n,d))$. For p odd a similar argument holds. We summarize the above discussion with the following theorem.

Theorem 1 *The complexity of the zone $Z_\sigma(H)$ of an algebraic surface σ of bounded degree and co-dimension p in the arrangement $\mathcal{A}(H)$ of n hyperplanes is $O(n^{d-\lceil p/2 \rceil})$ for p even and $O(n^{d-\lceil p/2 \rceil}\log n)$ for p odd.*

This theorem includes the result of Aronov and Sharir for $p = 1$. The bounds are almost tight except for the logarithmic factor for odd co-dimension.

5 Acknowledgments

I want to thank Boris Aronov for helpful discussions.

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