# Primal canoes: optimal arrangements of segments

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**Abstract:** In an earlier paper we presented tight $7\frac{1}{2}n + O(1)$ upper and lower bounds on the common intersection of n double wedges. This answered an outstanding question concerning the (exact) space complexity of an optimal algorithm which computes all transversals (stabbing lines) of a set of segments in the Euclidean plane. The structure used in computing the lower bound implicitly defines an arrangement of segments in the plane which we call the primal canoe. We wish to emphasize the properties of the primal canoe, and to present a new construction which exhibits the same complexity. We also wish to provide examples of primal canoes.

## 1 Introduction

Let $S$ be a set of $n$ closed line segments in the Euclidean plane $E^2$. A line intersecting all segments of $S$ is called either a transversal or a stabbing line. There is a natural duality, $D : E^2 \leftrightarrow E^2$ : which maps point $(\pi_1, \pi_2)$ and line $y = \pi_1 x - \pi_2$; see [Ede87]. This map carries a segment $s$ onto a union of lines called the double wedge $D(s)$. If $s$ is non-vertical then $D(S)$ contains two extremal lines, intersecting at a point called the center of the wedge and corresponding to the endpoints of $S$. These lines determine four regions; these two regions which do not contain the vertical line form a double wedge. Each point in the double wedge corresponds to a stabbing line of the corresponding segment in the primal space. This observation was applied in [EMP+82] to obtain an optimal algorithm which computes all transversals of a set of $n$ segments in the Euclidean plane. The space complexity of this algorithm depends on the number of edges in the common intersection of double wedges corresponding to the input segments.

In [EMP+82] it was shown that this number is not greater than $8n - 4$ and an arrangements with $6n-2$ edges was given. In [EJS89] the optimal arrangements of double wedges was studied and the tight bounds of $7\frac{1}{2}n+O(1)$ were proven (see also [JS91]). In particular, the arrangement, called *canoe*, presented in [EJS89] exhibited a collection of wedges having maximal number of edges in the polygons bounding the common intersection of wedges. This number of edges was shown to be $15k-6$ for a canoe consisting of $2k$ wedges.

## 2 Primal canoes

We will turn our attention to primal space and the original problem of finding common stabbing lines. We wish to emphasize the properties of arrangements of line segments that maximize the number of tan-

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gent stabbing lines; we will call them *primal canoes*. The stabbing line is called *tangent* if it passes through exactly two segment endpoints. In particular, we will show a new construction and examples of primal canoes and we will derive the following:

**Theorem** *There is an optimal arrangement of $2k$ segments that admits $13k - 8$ tangent stabbing lines.*

The number stated in the above theorem is optimal up to a small additive constant. The lower bound is realized by a delicate and sophisticated construction.

There is a clear duality between stabbing lines and the number of edges in the polygons of the common intersection of double wedges (so-called *stabbing region*). To this end, consider that a construction that maximizes the number of edges in a collection of polygons also maximizes the number of vertices. Some vertices are shared by two polygons, which complicates counting them. Careful charging shows that there are $13k - 8$ such vertices in a canoe. A point is a vertex precisely when it belongs to exactly two of the lines determining the wedges, and is interior to all other wedges. In primal terms, these vertices correspond to stabbing lines passing through exactly two endpoints of the collection of segments. This shows that there are at most $13k - 8$ tangent transversals through the $4k$ endpoints of the segments.

Since the canoe has the property that any two wedges have common interior, we see that the segments in the primal space must be pairwise intersecting. Thus, of the $13k - 8$ tangent transversals, $2k$ are simply embeddings of the segments.

There are several connections of our work to problems in combinatorial geometry. For example, a problem which relates to our results is investigated in [ELSS73] and [EW85]: given $2k$ points in $E^2$, how many partitions into two sets of respective $k$ points are there such that there is a line which separates the $k$ points of the one set from the $k$ points of the other set. The only bounds known on the maximal number of such partitions are $\Omega(k \log k)$ and $O(k\sqrt{k}/\log^* k)$; see [PSS89]. The results of this paper show that the number of such partitions is exactly $13k + O(1)$ if we require in addition that the $2k$ points are paired and a balanced partition is counted if and only if no pair is contained in either set.

Another combinatorial problem in geometry which relates to our results was considered in [KLZ85,ES87]. They studied the number of permutations in which a line can intersect $n$ non-intersecting segments in $E^2$.

# 3 Realization of canoes

At the end of the abstract we give examples which demonstrate both dual and primal canoes for small values of $k$. The new arrangements of wedges and line segments are called *asymmetric canoes* and *asymmetric primal canoes*, respectively. The illustrations were prepared using a system allowing the visualization of the duality process. It permits simultaneous viewing and manipulation of structures in primal and dual spaces. The user is permitted to independently control magnification and aspect ratios of the windows holding the two scenes (primal and dual), which is crucial due to the widely varying scales of features, all of which must be precisely located. For example, the portion shown of the 3-asymmetric canoe is roughly 1800 units wide and 1800 units high, while that of the primal canoe is roughly 2000 units wide and 6 units high.

The most recently added features of each inductive step are denoted by dashed lines, with corresponding features in the same style. The one exception to this rule is the 3-asymmetric canoe. Due to its scale, not all features of the 3-asymmetric primal canoe can be shown. The finely dashed line running from center left margin, which is a leg of the first wedge placed, has its center about 70 inches to the left of the margin. This center's placement is controlled by the point of intersection of the two nearly parallel lines appearing just below the finely dashed line. Similarly, the segment projecting downwards from the 3-asymmetric canoe has been clipped; to give an idea of its scale, it is roughly 400 times longer than the nearly horizontal segment in this primal canoe.

Note also that the primal canoes cannot be comprehended at any single scale. At any scale where the endpoints of the largest features are visible, the smallest features are not. For example, in the 4-canoe, the longest segment is roughly 100 times as long as the shortest.

## References


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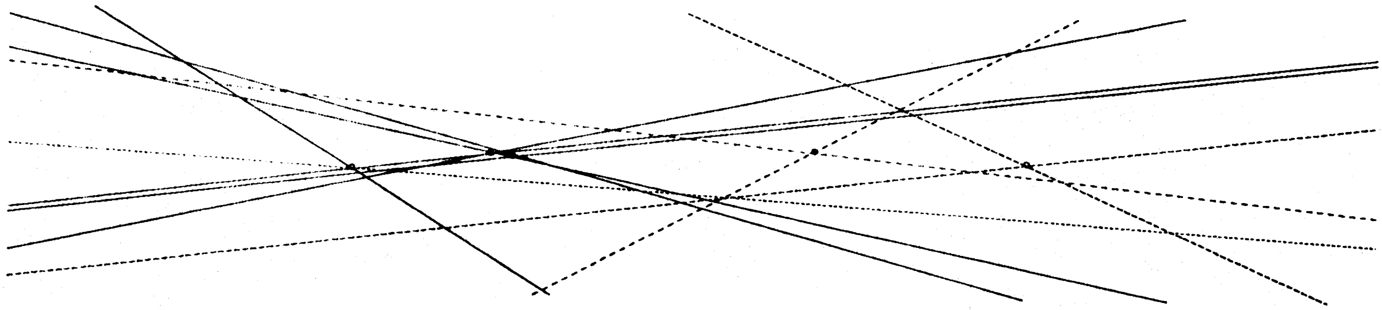
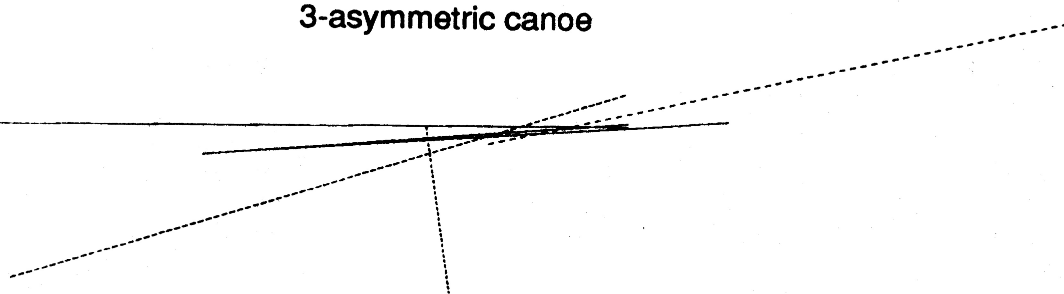
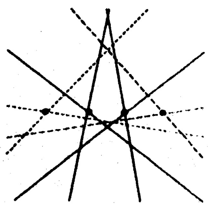


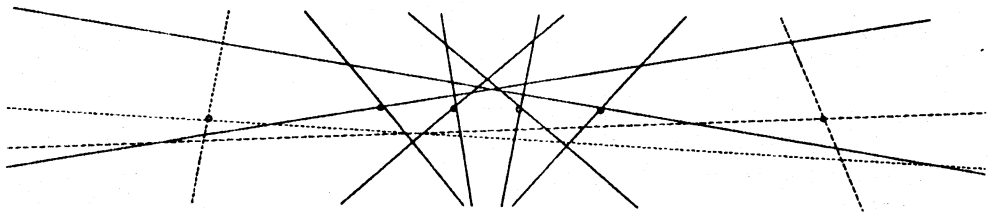
3-asymmetric canoe



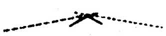
3-asymmetric primal canoe



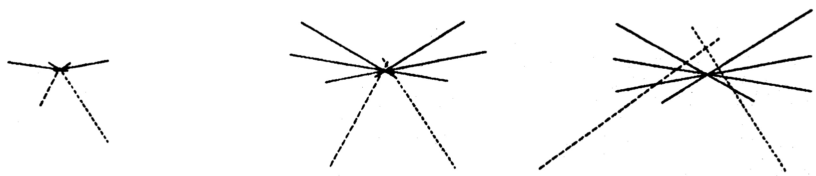
2-canoe



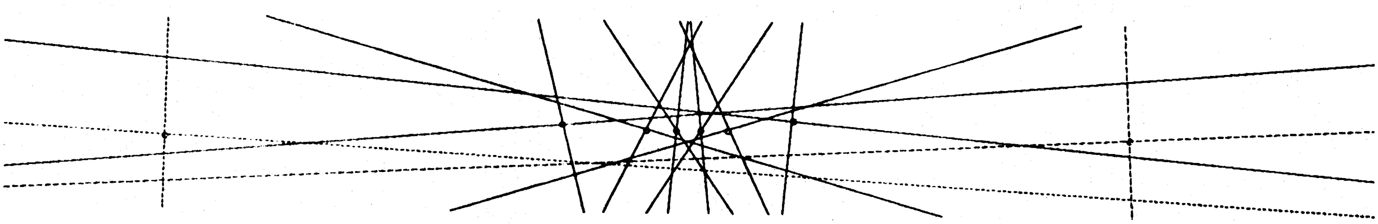
3-canoe



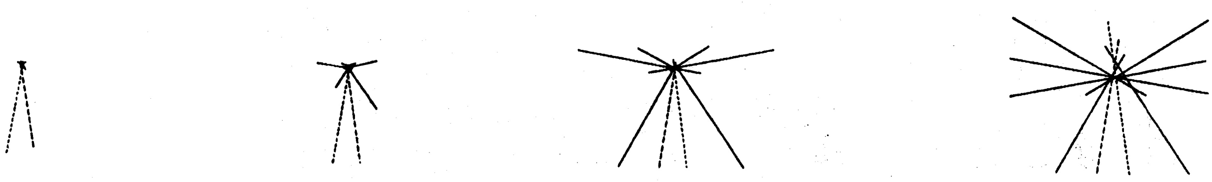
2-primal canoe



3-primal canoe -- increasing magnifications



4-canoe



4-primal canoe -- increasing magnifications