Shattering Configurations of Points With Hyperplanes

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Abstract. An arrangement of hyperplanes $\mathcal{A}$ in $\Re^d$ is said to *shatter* a point set $\mathcal{O}$ if each point of $\mathcal{O}$ is contained within the interior of its own cell of $\mathcal{A}$. In this paper, we investigate the number of hyperplanes required by an arrangement that shatters a set of n points in general position. We show that such sets can require between $\Omega(\sqrt[d]{n})$ and $\Omega(n)$ shattering hyperplanes. We also provide an algorithm that finds a linear-size shattering for such sets. The shattering produced exceeds the requirement of the worst known examples by at most a constant that depends only on d. We also give some results for when the points are in convex position.

1 Introduction

An arrangement of hyperplanes $\mathcal{A}$ in $\Re^d$ is said to *shatter* a point set $\mathcal{O}$ if each point of $\mathcal{O}$ is contained within the interior of its own cell of $\mathcal{A}$. Throughout this paper, we assume that point sets are finite and do not contain duplicate points.

Shattering problems were first investigated by [FMP90]. Their main results were showing that finding a minimum-cardinality shattering is NP-Complete for $d \geq 2$, and providing non-trivial algorithms for shattering polygonal objects in the plane and polyhedral objects in $\Re^d$. They also described how shattering problems are closely related to questions of stabbing and separability, which had been previously studied.

In this paper, we investigate the number of shattering hyperplanes required by different classes of point sets in $\Re^d$ (e.g. points in general position, and points in convex position). For each class C of interest, we answer the following combinatorial question: what are the minimum and maximum number of shattering hyperplanes required by a set of n points in C? This question was answered by [FMP90] for sets of points in $\Re^d$. We summarize their results in the following theorem.

Theorem 1: Let $\mathcal{O}$ be a set of n points in $\Re^d$.

1. $\mathcal{O}$ requires $\Omega(\sqrt[d]{n})$ shattering hyperplanes, and this bound is tight for some sets. The exact bounds for $d = 1, 2,$ and 3 are respectively $n-1$, $\left\lceil \frac{1}{2}(\sqrt{8n-7}-1) \right\rceil$, and $\left\lceil \frac{-5}{3R} + R \right\rceil$ where $R = \sqrt[3]{3n-3+\sqrt{\frac{125}{27}+(3-3n)^2}}$.
2. $\mathcal{O}$ can be shattered by $n-1$ parallel hyperplanes (determined in $O(n \log n)$ time). If all the points of $\mathcal{O}$ are collinear, then $n-1$ shattering hyperplanes are required.

Theorem 1 shows that a set of n points in $\Re^d$ can require between $\Omega(\sqrt[d]{n})$ and $n-1$ shattering hyperplanes. In Section 2, we show that the upper limit remains $\Omega(n)$, if the points required to be in general position (no subset of $d+1$ points lies on a common hyperplane) or general convex position. However when the points are in general convex position, the lower limit becomes $\Omega(\sqrt[d-1]{n})$. The exact upper limits are unknown for $d \geq 3$, but we give examples of sets that provide lower bounds. Upper bounds follow from the algorithm in Section 3 that produces a linear-size shattering for a set of n points in general position. The shattering produced exceeds the requirement of the worst known examples by at most a constant that depends only on d.

2 Combinatorial Results

In this section, we show that there are sets of n points in $\Re^d$ in general position that require $\Omega(n)$ shattering hyperplanes (see Theorem 2). Even if we further restrict the points to lie on the surface of a sphere ($d \geq 2$), there are still sets that require $\Omega(n)$ shattering hyperplanes (see Theorems 4 and 6 for $d \geq 3$, and observe that in $\Re^2$, every set of points on a circle requires $\Omega(n)$ shattering lines). Our Theorems are stated for $n \geq d+1$, because sets with fewer points require a more complicated definition of general position. However, our theorems still hold for such sets, since it is easy show that any set with $n \leq d$ points requires $\lceil \log_2 n \rceil$ shattering hyperplanes.

Theorem 2: In $\Re^d$, for any $n \geq d+1$, there exists a set $\mathcal{O}$ of n points in general position, such that any shattering of $\mathcal{O}$ requires $\left\lceil \frac{n-1}{d} \right\rceil$ hyperplanes if d is odd, and $\left\lceil \frac{n}{d} \right\rceil$ hyperplanes if d is even.

Proof: This is true for any set in $\Re^1$, so assume $d \geq 2$. Let $\mathcal{O} = \{p_1, p_2, \ldots, p_n\}$ be $n \geq d+1$ points in $\Re^d$

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on the moment curve with $p_i = (t_i, t_i^2, \ldots, t_i^d)$ and $t_1 < t_2 < \ldots < t_n$. Observe that $\mathcal{O}$ is the vertex set of a cyclic polytope, so the points must lie in general convex position. Consider a pair of neighboring points p_i and p_{i+1}. Any hyperplane that shatters the pair must intersect the piece of the moment curve between p_1 and p_n, since otherwise the pair would be in the same connected region. There are $n-1$ pairs, and any hyperplane can intersect the moment curve at most d times (Lemma 6.4 of [Ede87]). Therefore, $\lceil \frac{n-1}{d} \rceil$ hyperplanes are required to shatter $\mathcal{O}$. When d is even and $n \geq d+2$, any hyperplane h that intersects the moment curve d times between p_1 and p_n cannot also intersect the line segment $\overline{p_1 p_n}$ (Lemma 3). Therefore, $\lceil \frac{n}{d} \rceil$ hyperplanes are required. ∎

Lemma 3: Given $d+2$ points $p_0, p_1, \ldots, p_d, p_{d+1}$ in $\Re^d$ on the moment curve C with $p_i = (t_i, t_i^2, \ldots, t_i^d)$ and $t_0 < t_1 < \ldots < t_d < t_{d+1}$. Let π be the hyperplane defined by $p_1, \ldots, p_d$, and let $\overline{p_0 p_{d+1}}$ be the line segment between p_0 and p_{d+1}. Then $\pi \cap \overline{p_0 p_{d+1}} = \emptyset$ if and only if d is even.

Proof: Observe that π divides C into $d+1$ pieces $C_0, C_1, \ldots, C_d$, since there are d distinct single roots between π and C. Notice that p_0 lies on C_0, and p_{d+1} lies on C_d. The even indexed pieces lie in π^a, the closed half-space "above" π. Similarly, the odd indexed pieces lie in π^b, the closed half-space "below" π. Assume d is even. This implies that C_0 and C_d lie in π^a. Notice that both p_0 and p_{d+1} lie in the interior of π^a, since there can only be d intersections between π and C. Thus, $\overline{p_0 p_{d+1}}$ lies in the interior of π^a. If d is odd, C_0 and C_d lie in opposite half-spaces, so $\overline{p_0 p_{d+1}}$ must intersect π. ∎

R. Connelly provided the original proof that $\pi \cap \overline{p_0 p_{d+1}} = \emptyset$ when d is even, using a different approach.

Theorem 4: In $\Re^3$, for any $n \geq d+1$, there exists a set $\mathcal{O}$ of n points in general position on the surface of a sphere, such that $\lceil \frac{n-1}{3} \rceil$ planes are required to shatter $\mathcal{O}$.

Proof: Centrally project the positive half-parabola onto the surface of the unit sphere. The intersection between a plane and the surface of the sphere is a circle, so a plane intersects the projected curve in at most three points (Lemma 5). Therefore, any set of n points on the projected curve requires $\lceil \frac{n-1}{3} \rceil$ shattering planes. ∎

Lemma 5: In $\Re^2$, any circle $(x_1 - a)^2 + (x_2 - b)^2 = r^2$ can intersect the half parabola $x_2 = x_1^2, x_1 > 0$ at most three times.

Proof (P. Chew): Substituting for x_2 yields $(x_1 - a)^2 + (x_1^2 - b)^2 = r^2$. This simplifies to $x_1^4 + (1 - 2b)x_1^2 - 2ax_1 + (a^2 + b^2 - r^2) = 0$. The four roots must sum to zero, since the x_1^3 coefficient is zero. Therefore, there can be at most three positive real roots. ∎

Theorem 6: In $\Re^d$, $d \geq 3$, there exists a set $\mathcal{O}$ of n points in general position on the surface of the sphere, such that $\lceil \lfloor \frac{n}{3} \rfloor \frac{2}{d} \rceil$ hyperplanes are required to shatter $\mathcal{O}$.

Proof (R. Connelly, J. Mitchell): Choose $\lfloor \frac{n}{3} \rfloor$ locations on the surface of the sphere so that they lie in general position. At each location, place three points that form a triangle. Each triangle requires two shattering hyperplanes. The triangles are chosen small enough so that no hyperplane intersects more than d of them. Therefore, $\lceil \lfloor \frac{n}{3} \rfloor 2 \frac{1}{d} \rceil$ shattering hyperplanes are required. ∎

If $\lfloor \frac{n}{d} \rfloor$ $(d-1)$-simplices are used instead of the triangles, then a simple analysis yields the inferior $\lceil \lfloor \frac{n}{d} \rfloor \lceil \frac{d}{2} \rceil \frac{1}{d} \rceil$ bound. A more careful analysis may yield a superior bound.

Theorems 4 and 6 showed that in $\Re^d$, $d \geq 3$, there are sets of n points on the surface of a sphere that require $\Omega(n)$ shattering hyperplanes. However, the general lower bound for such sets is $\Omega(\sqrt[d-1]{n})$ shattering hyperplanes. This bound is derived from the complexity of a spherical zone in an arrangement of r hyperplanes, which is at worst the complexity of the the zone of the hyperplane at infinity ($O(r^{d-1})$, see Theorem 5.4 of [Ede87]). Therefore, at least $\Omega(\sqrt[d-1]{n})$ hyperplanes are required to shatter $\mathcal{O}$ (solve $n = O(r^{d-1})$).

3 Algorithmic Results

In Section 2, we provided examples of point sets in general position that require $\Omega(n)$ shattering hyperplanes. In this section, we give an algorithm to find a shattering for any set of points in general position that uses at most a constant more hyperplanes than the worst known examples require (see Theorem 7). The constant is zero for $d \leq 2$. We also provide a second algorithm that matches the worst-case bounds for the special case in $\Re^3$ when the points are the vertices of a cyclic polytope (see Theorem 10).

Theorem 7: Let $\mathcal{O}$ be a set of n points in $\Re^d$, $d \geq 2$. If $n \geq 2^{\lceil \log_2 d \rceil}$, then $\mathcal{O}$ can be shattered by $\left\lceil \frac{n - 2^{\lceil \log_2 d \rceil}}{d} \right\rceil + \lceil \log_2 d \rceil$ hyperplanes. In $\Re^3$, if $n \geq 8$, $\mathcal{O}$ can be shattered by $\lceil \frac{n+1}{3} \rceil$ planes. Otherwise, $\lceil \log_2 n \rceil$ hyperplanes suffice.

Proof: Let $q = \lceil \log_2 d \rceil$. Assume that $n \geq 2^q$. We say that a hyperplane h *bisects* a point set $\mathcal{S}$ if h does not contain any points of $\mathcal{S}$ and the two subsets of $\mathcal{S}$

produced by h differ in size by at most one. We begin by finding a hyperplane that bisects $\mathcal{O}$, producing subsets $\mathcal{O}_1^1$ and $\mathcal{O}_2^1$. Next, we find a ham-sandwich hyperplane that bisects each $\mathcal{O}_i^1$, producing $\mathcal{O}_1^2, \ldots, \mathcal{O}_4^2$ (see Lemma 8). We continue this process until we have determined q ham-sandwich hyperplanes that split $\mathcal{O}$ into $\mathcal{O}_1^q, \ldots, \mathcal{O}_{2^q}^q$, such that the first $n - 2^q \lfloor \frac{n}{2^q} \rfloor$ subsets have $\lceil \frac{n}{2^q} \rceil$ points and the remainder have $\lfloor \frac{n}{2^q} \rfloor$ points. This process can be represented as a balanced binary tree, with each node corresponding to a subset of $\mathcal{O}$. The root (level 0) is $\mathcal{O}$. Level k of the tree has the 2^k subsets $\mathcal{O}_1^k, \ldots, \mathcal{O}_{2^k}^k$, each with either $\lceil \frac{n}{2^k} \rceil$ or $\lfloor \frac{n}{2^k} \rfloor$ points.

We will continue to bisect subsets until every point of $\mathcal{O}$ is in a singleton set, which corresponds to filling out the tree. Each singleton will be a leaf on level $\lceil \log_2 n \rceil - 1$ or $\lceil \log_2 n \rceil$. Each non-leaf node corresponds to an intermediate subset that must be bisected by some hyperplane. Each subtree rooted at a level q node has $\lceil \frac{n}{2^q} \rceil - 1$ or $\lfloor \frac{n}{2^q} \rfloor - 1$ non-leaf nodes which must be bisected. So $n - 2^q$ non-leaf nodes remain to be bisected.

Let $\mathcal{A}$ be the collection of active subsets. Initially $\mathcal{A} = \{\mathcal{O}_1^k, \ldots, \mathcal{O}_{2^k}^k\}$. We choose a ham-sandwich hyperplane that bisects the d largest subsets of $\mathcal{A}$. We then update $\mathcal{A}$ and repeat this process until $\mathcal{A}$ contains the n singletons. Except possibly for the last hyperplane chosen, it is always possible to find d subsets of $\mathcal{A}$ to bisect, since at any step only two levels of the tree can contain active subsets, and each full level has $2^q \geq d$ nodes. We use $\lceil \frac{n-2^q}{d} \rceil$ additional hyperplanes to complete the shattering of $\mathcal{O}$. Therefore, $\mathcal{O}$ can be shattered by $q + \lceil \frac{n-2^q}{d} \rceil$ hyperplanes.

In $\Re^3$, three planes can be found that divide $\mathcal{O}$ into eight pieces whose sizes differ by at most one (Lemma 9). This improves the bound on the number of planes required to $3 + \lceil \frac{n-8}{3} \rceil = \lceil \frac{n+1}{3} \rceil$. It is unknown if a similar result holds in $\Re^4$. In $\Re^d$, $d \geq 5$, no such result is possible.

If $n < 2^q$, then apply our initial process to determine $\lceil \log_2 n \rceil$ ham-sandwich hyperplanes that shatter $\mathcal{O}$. ■

In $\Re^2$, a ham-sandwich hyperplane can be determined in $O(n)$ time (see Theorem 14.6 of [Ede87]), so the algorithm runs in $O(n^2)$ time. In $\Re^d$, $d \geq 3$, the algorithm requires $O(n^{d+1})$ time, since $O(n^d)$ time is required to determine a ham-sandwich hyperplane. However in $\Re^3$, $O(n^7)$ time is required if the initial procedure that splits $\mathcal{O}$ into eight equal pieces is used.

Lemma 8: Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_d$ be sets of points in $\Re^d$, such that all the points in $\mathcal{O} = \bigcup_i \mathcal{O}_i$ are in general position. There exists a hyperplane h that does not contain any points of $\mathcal{O}$ and simultaneously bisects each $\mathcal{O}_i$, $1 \leq i \leq d$.

Proof: By Theorem 4.7 of Edelsbrunner[Ede87], there exists a hyperplane h', such that neither open half-space defined by h' contains more than half the points of any set $\mathcal{O}_i$, with h' containing any remaining points. Observe that h' can contain up to d points of $\mathcal{O}$, since all the points are in general position. We will show by induction on d, that it is possible to perturb h' to produce the desired hyperplane h.

In $\Re^1$, the point h' might be in $\mathcal{O}_1$. If so, let p be the closest neighbor in $\mathcal{O}_1$ to h'. Choose h to be any point in the open interval between h' and p.

In $\Re^2$, the line h' may contain two points p_1, p_2 of $\mathcal{O}$. If so, let g be a point in the open interval between p_1 and p_2 on h'. We produce h by slightly rotating h' about g, such that no other point of $\mathcal{O}$ is intersected. If only one point lies on h', then we produce h by slightly translating h'.

Assume that the lemma is true for $\Re^{d-1}$. In $\Re^d$, it is possible for d points to lie on the hyperplane h'. Let $i_1, i_2, \ldots, i_k$ be the indices of sets $\mathcal{O}_i$ that contain two or more points on h', and let $\mathcal{S}_j = \mathcal{O}_{i_j} \cap h$. Let $\mathcal{S} = \mathcal{O} \cap h$ and $\overline{\mathcal{S}} = \mathcal{S} \setminus \bigcap_j \mathcal{S}_j$. Observe that $k \leq \lfloor \frac{d}{2} \rfloor$. By the induction hypothesis, there is a $(d-1)$-hyperplane g in $\mathcal{S}$ that does not contain any points of $\mathcal{S}$ and bisects $\mathcal{S}_1, \ldots, \mathcal{S}_k, \overline{\mathcal{S}}$. We produce h by slightly rotating h' about g. ■

Lemma 9: Let $\mathcal{O}$ be a set of points in general position in $\Re^3$. There exist three planes h_1, h_2, h_3 that do not contain any points of $\mathcal{O}$, and split $\mathcal{O}$ into eight pieces whose sizes differ by at most one.

Proof: Determine a plane h_1 that does not contain any points of $\mathcal{O}$ and bisects $\mathcal{O}$, producing $\mathcal{O}_1$ and $\mathcal{O}_2$. By Theorem 4.12 of Edelsbrunner[Ede87], there exist planes h_2', h_3' that together divide each $\mathcal{O}_i$ into four pieces whose sizes differ by at most one. Each plane h_2', h_3' may contain two points. We produce h_2 and h_3 by slightly rotating h_2' and h_3', so that the sizes of the eight pieces differ at most by one. ■

In $\Re^3$, if the points of $\mathcal{O}$ define a cyclic polytope, then we can shatter them slightly more efficiently. Theorem 10 shows it is possible to match the bound given in Theorem 2.

Theorem 10: Given a set $\mathcal{O} = \{p_1, p_2, \ldots, p_n\}$ of n points in $\Re^3$ on the moment curve C. If $n \geq 11$, then there exist $\lceil \frac{n-1}{3} \rceil$ planes that shatter $\mathcal{O}$.

Proof: We will prove the theorem by induction. We assume that $p_i = (t_i, t_i^2, t_i^3)$, and $0 < t_1 < t_2 < \ldots < t_n$.

Assume that $\mathcal{O} = \{p_1, p_2, \ldots, p_{13}\}$. Let π^1 be the plane that intersects C once on each of the following open arcs: (t_3, t_4), (t_6, t_7), and (t_8, t_9). π^1 splits $\mathcal{O}$ into

n Distinct Real Points in	Number of Shattering Hyperplanes minimum required	worst known example	achievable algorithmically
$\Re^1$, $\Re^d$:Collinear	$n-1$	$n-1$	$n-1$
$\Re^2$:General Convex Position	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$
$\Re^2$:General Position	$\lceil \frac{1}{2}(\sqrt{8n-7}-1) \rceil$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$
$\Re^3$:Cyclic Polytope	$\lceil \frac{n-1}{3} \rceil$	$\lceil \frac{n-1}{3} \rceil$	$\lceil \frac{n-1}{3} \rceil$
$\Re^3$:General Position on the surface of a sphere	$\Omega(\sqrt{n})$	$\lceil \frac{n-1}{3} \rceil$	$\lceil \frac{n+1}{3} \rceil$
$\Re^3$:General Position	$\lceil \frac{-5}{3R} + R \rceil$, where $R = \sqrt[3]{3n-3+\sqrt{\frac{125}{27}+(3-3n)^2}}$	$\lceil \frac{n-1}{3} \rceil$	$\lceil \frac{n+1}{3} \rceil$
$\Re^4$:Cyclic Polytope	$\lceil \frac{n}{4} \rceil$	$\lceil \frac{n}{4} \rceil$	$\lceil \frac{n}{4} \rceil$
$\Re^d$:General Position on the surface of a sphere	$\Omega(\sqrt[d-1]{n})$	$\lceil \frac{2n}{3d} \rceil$	$\left\lceil \frac{n-2^{\lceil \log_2 d \rceil}}{d} \right\rceil + \lceil \log_2 d \rceil$
$\Re^d$:General Position, d even	$\Omega(\sqrt[d]{n})$	$\lceil \frac{n}{d} \rceil$	$\left\lceil \frac{n-2^{\lceil \log_2 d \rceil}}{d} \right\rceil + \lceil \log_2 d \rceil$
$\Re^d$:General Position, d odd	$\Omega(\sqrt[d]{n})$	$\lceil \frac{n-1}{d} \rceil$	$\left\lceil \frac{n-2^{\lceil \log_2 d \rceil}}{d} \right\rceil + \lceil \log_2 d \rceil$

Table 1: Minimum and maximum number of shattering hyperplanes required by different classes of point sets.

$\{p_1, p_2, p_3, p_7, p_8\}$ and $\{p_4, p_5, p_6, p_9, p_{10}, p_{11}, p_{12}, p_{13}\}$. Let π^2 be the plane that intersects C once on each of the following open arcs: (t_2, t_3), (t_4, t_5), and (t_{10}, t_{11}). π^2 further splits $\mathcal{O}$ into $\{p_1, p_2, p_7, p_8\}$, $\{p_3\}$, $\{p_4, p_{11}, p_{12}, p_{13}\}$, and $\{p_5, p_6, p_9, p_{10}\}$. Let π^3 be the plane that intersects C once on each of the following open arcs: (t_1, t_2), (t_7, t_8), and (t_{12}, t_{13}). π^3 further splits $\mathcal{O}$ into $\{p_1, p_8\}$, $\{p_2, p_7\}$, $\{p_3\}$, $\{p_4, p_{13}\}$, $\{p_5, p_6\}$, $\{p_9, p_{10}\}$, and $\{p_{11}, p_{12}\}$. Let π^4 be the plane that intersects C once on each of the following open arcs: (t_5, t_6), (t_9, t_{10}), and (t_{11}, t_{12}). π^4 completes the shattering of $\mathcal{O}$. Therefore, there exist four planes that shatter $\mathcal{O}$ for $11 \leq n \leq 13$.

Assume that the theorem is true for $n = k-3 \geq 11$. Let $\mathcal{O}$ be a set of k points, and $\mathcal{O}' = \mathcal{O} \backslash \{p_2, p_5, p_8\}$. By the induction hypothesis, $\mathcal{O}'$ can be shattered by Π, a set of $\left\lceil \frac{(k-3)-1}{3} \right\rceil$ planes. Consider the arrangement of planes in Π, along with the points of $\mathcal{O}$. The three cells that contain p_2, p_5, and p_8 are the only ones with two points. Every other cell contains at most one point. So, one additional plane is sufficient to complete the shattering of $\mathcal{O}$. Therefore, there exist $\lceil \frac{k-4}{3} \rceil + 1 = \lceil \frac{k-1}{3} \rceil$ planes that shatter $\mathcal{O}$. ∎

A similar result holds for cyclic polytopes in $\Re^4$. The base case is shattering between 13 and 16 points with four hyperplanes. These results should generalize to $\Re^d$.

4 Conclusion

We conclude by summarizing our results in Table 1. Observe that there are still small gaps to be closed between the bounds for the maximum number of hyperplanes required in $\Re^d$, $d \geq 3$. A larger gap exists for points that lie on the surface of a sphere in $\Re^d$, $d \geq 4$.

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