On the Maximal Number of Edges of Digital Convex Polygons Included into a Square Grid

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Abstract

Let $e(m)$ denote the maximal number of edges of a digital convex polygon included into an $m \times m$ square area of lattice points and let $s(n)$ denote the minimal (side) size of a square in which a convex digital polygon with n edges can be included. We prove that

$$e(m) = \frac{12}{(4\pi)^{2/3}}\, m^{2/3} + O(m^{1/3} \log m)$$

$$s(n) = \frac{2\pi}{12^{3/2}}\, n^{3/2} + O(n \log n)$$

Digital convex polygons are those convex polygons, all the vertices of which have integer coordinates. We investigate the relationships between the number $n(P)$ of edges of a digital convex polygon P and side length $m(P)$ of a minimal digital square (with edges parallel to coordinate axes), in which P might be included. Let $s(n)$ denote the minimal $m(P)$ over all P with $n(P) = n$. Similarly, let $e(m)$ denote the maximal $n(P)$, over all P with $m(P) = m$.

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There is a sequence $P(t)$ of digital convex polygons, such that the increasing sequences $m(t) = m(P(t))$ and $n(t) = n(P(t))$ of natural numbers correspond to each other in the above two optimization problems. In other words, $m(t) = s(n(t))$ and $n(t) = e(m(t))$. Moreover, it is shown that the polygons $P(t)$ are the unique optimal solutions for the numbers $m(t)$, respectively $n(t)$.

We proceed with a sketch of the way in which the considered optimization problems are solved:

Let

$$bd(s) = |x_1 - x_2| + |y_1 - y_2| \ .$$

denote the block (Manhattan) distance associated with the line segment $s = \{(x_1, y_1), (x_2, y_2)\}$.

Lemma 1 *The sum of block distances associated to all the edges of a convex digital polygon P is equal to the perimeter of the minimal rectangle with sides parallel to the coordinate axes, which includes P.*

A consequence of this lemma is the following inequality:

$$m \geq \frac{1}{4} \sum_{e \in P(m)} bd(e) \ ,$$

where $P(m)$ denotes the digital convex polygon included into $m \times m$–grid with the maximal possible number of edges (i.e., with $e(m)$ edges).

The definition of $P(m)$ requires that the sum on the left-hand side of the above inequality has the maximal possible number of summands. Consequently, since the sum is bounded from above with the constant $4 * m$, we should keep these summands as small as possible.

We associate the fraction $|x_1 - x_2|/|y_1 - y_2|$ to each summand $|x_1 - x_2| + |y_1 - y_2|$ of the above sum, where (x_1, x_2) and (y_1, y_2) are two consecutive vertices of $P(m)$.

Since three mutually parallel edges cannot exist in a convex polygon, it follows that each of the above fractions cannot appear more than four times as the fraction associated to a summand of the above sum.

These facts are sufficient for completing a greedy argument, which gives the optimality of the polygons $P(t)$.

The explicit expressions for $m(t)$ and $n(t)$ are:

$$m(t) = 1 + \sum_{\substack{p \perp q \\ p+q \leq t}} p =$$

$$\sum_{1 \leq n \leq t} s * \phi(s) = \frac{2t^3}{\pi^2} + O(t^2 \log t)$$

The last equality is given in [1].

$$n(t) = 4 + 4 * \sum_{\substack{p \perp q \\ p+q \leq t}} 1 = 4 + 4 * \sum_{s=1}^{t} \phi(s) = \frac{12t^2}{\pi^2} + O(t \log t)$$

The last equality is given in [2].

($\phi(s)$ denotes the Euler function, the number of integers between 1 and s which are relatively prime with s)

The asymptotic estimations for functions $s(n)$ and $e(m)$, for arbitrary values of m and n are derived on the basis of the previous expressions and the fact that the functions $s(n)$ nad $g(m)$ are monotonously increasing functions.

The main result of this paper is the following theorem:

Theorem 1

$$e(m) = \frac{12}{(4\pi^2)^{1/3}} m^{2/3} + O(m^{1/3} \log m)$$

$$s(n) = \frac{2\pi}{12^{3/2}} n^{3/2} + O(n \log n) .$$

Remark. The empirically obtained coefficients of the leading members in the asymptotic expressions for $s(n)$ and $e(m)$ were 0.1507 and 3.53 respectively ([3]). Their more exact values, which are derived from Theorem 1., up to four decimal places, are 0.1511 and 3.5242 respectively.

References

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[3] Voss, K., Klette, R., *On the maximal number of edges of a convex digital polygon included into a square*, Pocitace a umela inteligencia, Vol.1., Bo.6, Dec. 1982, 549 - 558.