Two-Dimensional Grid Spanners

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Abstract

A t-spanner of a graph is a subgraph in which every two nodes that were connected by an edge in the original graph are connected by a path of at most t edges in the subgraph. We present t-spanners of minimum average degree for infinite 2-dimensional grids. The minimality of degree is shown by an interesting connection with polyomino tiling.

1 Introduction

It is often the case in computational geometry that geometric problems are solved by first transforming the problem to a graph-theory problem, and then solving this problem. In this paper, we outline a problem's solution which runs counter to this trend: we solve a networking (graph) problem by transforming it to a geometric problem.

Many network topologies have been proposed for parallel computers; foremost among these are the pyramid, hypercube, and the multidimensional grid (mesh). One drawback to these structures is their relatively high degree at each vertex. In this paper, we consider *spanners* as a method of dealing with this problem. Spanners are substructures which have the property that the edges removed from the original structures are effectively replaced by a short path in the substructure. This is distinct from a related technique in the literature, graph embedding, in which a "logical network" is mapped onto another "host network". (See, for example, [1].) We are interested in the particular case of finding spanners for 2-dimensional grids. Spanners of pyramids, hypercubes, and multidimensional grids have been studied elsewhere [5] [7] [8].

A network is represented by a connected simple graph $G = (V, E)$. A spanning subgraph $S = (V, E')$ of G is a *t-spanner* if for any edge (a, b) of G there is a path from a to b in S with t or fewer edges [6]. The value t is known as *dilation* or *stretch-factor*. Our goal is to construct, for the two-dimensional grid, small dilation spanners with low average degree. As the grid is bipartite, we need consider only odd values of t. The average degree of the spanner S is denoted δ.

Another important measure of the quality of a spanner is *congestion*, that is, the amount of traffic expected across edges of S. Given an assignment of paths in S corresponding to edges in G, the congestion of an edge under that assignment in S is the number of paths that use that edge (including the edge itself). The *congestion* of S, denoted γ, is the maximum congestion of any edge in S, where γ is minimized over all assignments of paths. For example, removing a single edge from any complete graph K_n on $n \geq 3$ vertices gives a 2-spanner of K_n with $\gamma = 2$.

Peleg and Schäffer [6] investigated the existence and constructability of sparse spanners for various classes of graphs. In particular, they show that finding a minimum-edge 2-spanner in a given graph is NP-hard. Richards and Liestman [8] began the investigation of degree-constrained spanners by studying these substructures in pyramids. Liestman and Shermer [5] continued this study for hypercubes and mulitdimensional grids ([5] contains the full exposition of the results reported here). Peleg and Ullman [7] constructed a 3-spanner of the d-cube with average degree 12. Cai [2] has characterized those graphs

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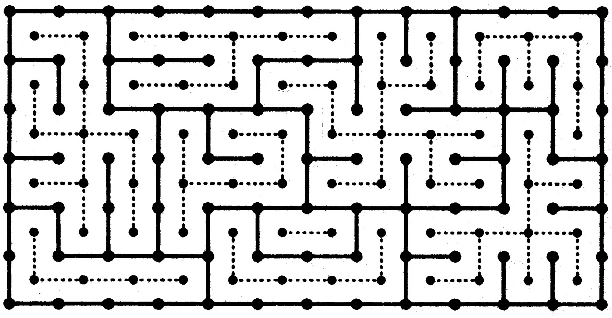

Figure 1: A finite grid spanner and its skeleton

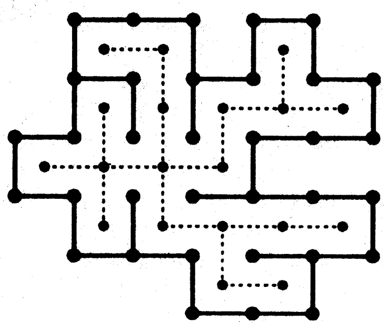

Figure 2: A tile, its skeleton, and induced subspanner

that have 2-spanners that are trees.

We show here how to construct t-spanners of infinite 2-dimensional grids with minimum average degree. This construction yields spanners with congestion t, which we show to be optimal for $t \equiv 3 (\text{mod } 4)$. By appropriate truncation of our infinite spanners, we can construct spanners for finite 2-dimensional grids with average degree and congestion approximating that of the infinite construction.

2 Lower Bound on $\delta(S)$

Throughout this section, we consider a t-spanner S of the two-dimensional grid $G_{(2)}$.

We define the ***dual*** $D_{(2)}$ of $G_{(2)}$ to be a graph with one vertex per unit square of $G_{(2)}$, with an edge between vertices of $D_{(2)}$ corresponding to squares of $G_{(2)}$ that share an edge. In this dual mapping, each edge e of $G_{(2)}$ has a corresponding dual edge $D(e)$ in $D_{(2)}$; $D(e)$ is that edge of $D_{(2)}$ which is between the vertices corresponding to the two squares of $G_{(2)}$ sharing the edge e.

We define the ***skeleton*** $SK(S)$ of the spanner S to be a graph with the vertex-set of $D_{(2)}$, and containing all dual edges $D(e)$ such that e is ***not*** in S. A finite grid spanner and its skeleton are shown in Figure 1.

Each connected component of the skeleton $SK(S)$ gives rise to a ***tile*** of S. A tile T is the collection of grid squares corresponding to the vertices of a component SK_i of $SK(S)$; SK_i is called the skeleton of tile T, and denoted $SK(T)$. Each tile is therefore a ***polyomino*** (connected set of grid squares). Tiling properties of polyominoes have been well studied (e.g., [3] [4]). We let N_T denote the number of squares in T, V_T denote the vertices of the grid which are on T, and S_T denote the subgraph of S induced by V_T. A tile, its skeleton, and its induced subgraph are shown in Figure 2. No tile skeleton will contain a cycle; such a cycle would imply a disconnected spanner.

Lemma 2.1 *Every tile T of a t-spanner S of a 2-dimensional grid has*

$$N_T \leq 4\lfloor (t+1)/4 \rfloor \lfloor (t+3)/4 \rfloor + 1.$$

Tiles realizing this bound consist of a centroid unit square surrounded by four $\lfloor (t+1)/4 \rfloor$ by $\lfloor (t+3)/4 \rfloor$ rectangles.

Lemma 2.2 *For any tile T in a spanner,* $\delta(SK(T)) = 2 - 2/N_T$.

Corollary 2.2a *For any spanner S of $G_{(2)}$ with all tiles having N squares,* $\delta(S) = 2 + 2/N$.

An interesting particular case of the previous corollary is a ***monohedral*** spanner: a spanner with all tiles identical. We will use monohedral spanners in our constructions of Section 3.

We can combine Lemmas 2.1 and 2.2 to get:

Corollary 2.2b *For any tile T in a t-spanner,*

$$\delta(SK(T)) \leq 2 - \frac{2}{4\lfloor (t+1)/4 \rfloor \lfloor (t+3)/4 \rfloor + 1}.$$

As there is some T with $\delta(SK(T))$ at least as large as $\delta(SK(S))$, the above bound also holds for $\delta(SK(S))$. This gives the following lower bound on the average degree of any t-spanner of $G_{(2)}$:

Theorem 2.3 *For any t-spanner S of $G_{(2)}$,*

$$\delta(S) \geq 2 + \frac{2}{4\lfloor (t+1)/4 \rfloor \lfloor (t+3)/4 \rfloor + 1}.$$

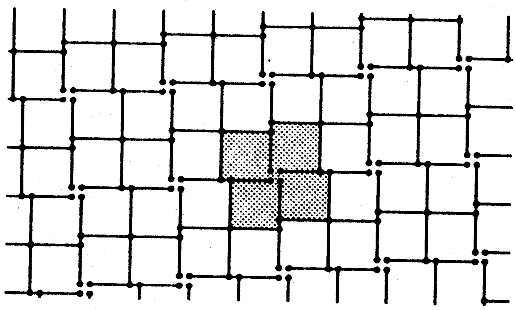

Figure 3: The tiling for $p = q$

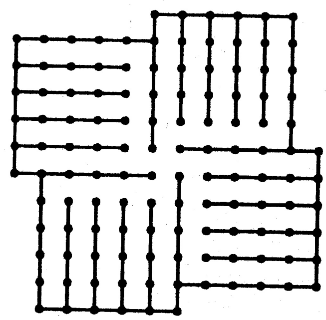

Figure 4: A tile with entire subspanner shown

3 δ-Optimal t-Spanners

We now show that the lower bound of Theorem 2.3 can be realized.

Theorem 3.1 *For any odd $t \geq 1$, there exist t-spanners of $G_{(2)}$ with*

$$\delta(S) = 2 + \frac{2}{4\lfloor (t+1)/4 \rfloor \lfloor (t+3)/4 \rfloor + 1}.$$

PROOF If $t = 1$, then $G_{(2)}$ is itself the spanner, with $\delta(S) = 4$. We henceforth consider only odd $t \geq 3$.

Let $p = \lfloor (t+1)/4 \rfloor$ and $q = \lfloor (t+3)/4 \rfloor$. Either $p = q$ or $p = q - 1$. In the former case, $t \equiv 1 \pmod 4$ and in the latter case, $t \equiv 3 \pmod 4$. By Corollary 2.2a, a tiling with all tiles having $4pq + 1$ squares will give us the desired average degree. Consider a tile T with $N_T = 4pq + 1$; such a tile consists of a centroid square connected to four p by q rectangles.

If $p = q$, then (barring enantiomorphs) the tile is of the shape shown shaded in Figure 3. The interiors of the large squares can be divided in any manner that creates no cycles in the skeleton and does not violate the dilation constraint; one such scheme is shown in Figure 4. We will later consider how to divide these squares to reduce congestion. A spanner of the desired average degree is constructed by tiling the plane using these tiles. Such a tiling, in fact the only one, is shown in Figure 3.

If $p = q - 1$, then there are six possible general tile shapes which are shown in Figure 5. The tile shapes of Figures 5c, d, e, and f always admit monohedral tilings of the plane; any of these tilings satisfies the theorem. For the special case when $t = 5$, there are three extra tile shapes, giving the nine tiles shown in Figure 6. Of these tiles, all but the large cross tile (Figure 6a) admit monohedral tilings. □

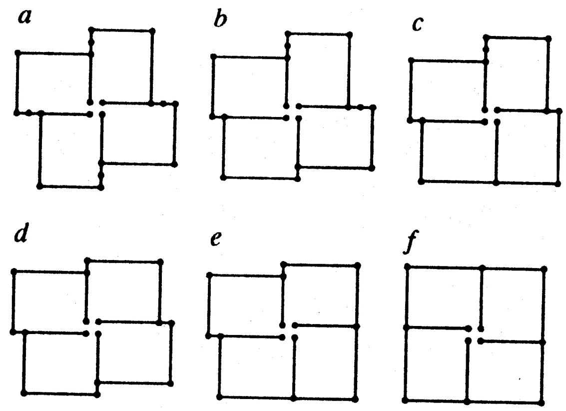

Figure 5: Possible tile shapes if $p = q - 1$

We call any spanner of $G_{(2)}$ which satisfies the previous theorem a *δ-optimal spanner* and any tile used in the construction in the proof a *δ-optimal tile.* Note that δ-optimal spanners may contain tiles which are not δ-optimal tiles, but only if such tiles are "infinitely outnumbered" by δ-optimal tiles.

4 Congestion on δ-Optimal Spanners

We now investigate congestion on δ-optimal spanners. The following two lemmas provide a quick visual method for determining the congestion on a δ-optimal tile.

Lemma 4.1 *For any internal edge e of a tile of any spanner S of a 2-dimensional grid G, the congestion on e is equal to one plus the distance in $SK(S)$ of the endpoints of $D(e)$.*

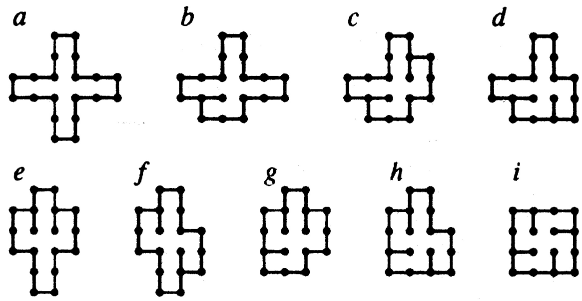

Figure 6: Tiles for $t = 5$

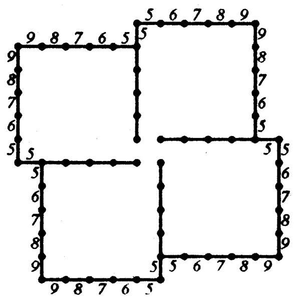

Figure 7: Minimum congestion on border edges

Given a δ-optimal tile T, we let z_T denote the centroid vertex of $SK(T)$. (The skeleton of any such tile must have exactly one centroid vertex.) For any border edge e on any tile T, we let $q(e)$ denote the vertex of $SK(T)$ dual to the square of T adjacent to e.

Lemma 4.2 *For any border edge e on any δ-optimal tile T, missing edges on T contribute congestion to e equal to the distance between $q(e)$ and z_T in $SK(T)$.*

Using the previous lemma, we can label the border edges of the δ-optimal tile shapes, with the least possible congestion that missing edges on the tile could contribute to that edge. Figure 7 shows this labelling for the 19-spanner tile shape. In general, this minimum contributed congestion will vary between $\lfloor (t+1)/4 \rfloor$ and $(t-1)/2$.

Lemma 4.3 *For every $t \equiv 3 \pmod 4$, $t \geq 3$, any δ-optimal t-spanner of $G_{(2)}$ must have $\gamma \geq t$.*

PROOF Consider a δ-optimal tile in the middle of a large area of δ-optimal tiles in the spanner; the spanner locally looks the tiling in Figure 3. Hence, border edges with minimum contributed congestion of $(t-1)/2$ from two different tiles coincide. Therefore, the spanner has congestion at least t. □

We believe that the $\gamma \geq t$ bound holds for $t \equiv 1 \pmod 4$ as well, except for the special case $t = 9$. Due to the large number of tiles and tilings, we have not been able to obtain a proof. We are able to produce δ-optimal t-spanners with $\gamma = t$:

Lemma 4.4 *For every odd $t \geq 1$, there exist δ-optimal t-spanners of $G_{(2)}$ with $\gamma = t$.*

PROOF We use the tilings given in the proof of Theorem 3.1, with tiles divided as in Figure 4. In the tiling, border edges with congestion $(t-1)/2$ from each of two tiles coincide, giving a congestion of t on each such edge; this is the maximum possible on a border edge. By Lemma 4.1, the internal edges will have congestion at most $2p$, where $p = \lfloor (t+1)/4 \rfloor \leq (t+1)/4$; Thus the maximum congestion is found on the external edges with congestion t. □

We can also construct a $\gamma = 7$ δ-optimal 9-spanner of $G_{(2)}$; a simple probabilistic argument shows that this is the smallest γ possible for such a spanner. However, since the tiling we use exists only for $t = 9$ and not for other $t \equiv 1 \pmod 4$, we believe this t-spanner with $\gamma < t$ to be a special case.

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