A PROPERTY OF CONVEX POLYGONS

(Extended Abstract)

By

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1 Introduction.-

A collection S of plane polygons is *well supported* if at least one side of a polygon in S is contained in the boundary of the convex closure of S. A line l *separates* a set A from a collection G of plane sets if A is contained in one of the closed halfplanes determined by l while every set in G is contained in the complementary closed halfplane.

Not every collection of $n \geq 3$ convex sets is well supported; in this article we prove that every collection of n convex polygons in the plane with pairwise disjoint interiors contains a well supported subcollection with at least $\lceil (3n+55)/54 \rceil$ sets.

In [5] H. Tverberg proved that for each positive integer k, there is a minimum interger f(k) such that for every collection F of f(k) or more plane compact convex sets with pairwise disjoint interiors, there is a line that separates one set in F from a subcollection of F with at least f(k) sets. K. Hope and M. Katchalski showed in [3] that $3k+1 \leq f(k) \leq 12k-11$.

In this article, we prove that if F is a collection in the plane of n convex polygonal regions, not necessarily bounded, with pairwise disjoint interiors, there is a side s of a region in F such that the line supporting s separates a region in F from a subcollection of F with at least $\lceil (3n+1)/54 \rceil$ sets. As a corollary we show that in every collection F of n compact plane convex sets with pairwise disjoint relative interiors, there are two sets A and B such that every line that separates A from B separates either A or B from a subcollection S of F with at least $\lceil (3n+1)/54 \rceil$ sets. This improves a bound given by E. Rivera-Campo in [4].

2 Preliminary Results.-

The following result was proved in [1]; see also [2].

Theorem .- Any collection of n compact, convex and pairwise disjoint sets in the plane may be covered with n disjoint convex polygons with a total of not more than 6n-9 sides. Furthermore, no more than 3n-6 distinct slopes are required.■

We adapt the proof given in [1] to obtain the following lemma.

Lemma 1.- Let $P = \{P_1, P_2, \ldots, P_n\}$ be a collection of $n \geq 3$ plane convex polygons (polygonal regions) with pairwise disjoint interiors. There exists a collection $R = \{R_1, R_2, \ldots, R_n\}$ of plane convex polygons (polygonal regions) with pairwise disjoint interiors such that:

1) For $i = 1, 2, \ldots, n$, $P_i \subset R_i$.
2) Each side of R_i contains a side of P_i.
3) The total number of sides among $R_1, R_2, \ldots, R_n$ is at most 9n-12.■

3 Main Results.-

Theorem 2.- If $P = \{P_1, P_2, \ldots, P_n\}$ is a collection of disjoint convex polygons in the plane, then P contains a well supported subcollection with at least $\lceil (3n+55)/54 \rceil$ sets.

Proof.- By Lemma 1, there is a collection $R = \{R_1, R_2, \ldots, R_n\}$ satisfying 1), 2) and 3). Let $S = \{ s_1, s_2, \ldots s_m \}$ be the set of all sides of the polygons in R.

For each s_j in S, let $R_{i(s_j)}$ and $H^+(s_j)$ denote the polygon in R that contains s_j and the closed half plane determined by s_j that contains $R_{i(s_j)}$, respectively. Define a bipartite graph G as follows: G has a vertex u_i for each polygon $R_i \in R$ and a vertex v_j for each side $s_j \in S$. There is an edge $u_i v_j$ in G if the polygon R_i is contained in $H^+(s_j)$.

Since any pair of polygons $\{ R_p, R_q \}$ is well supported, then there is at least one edge in G for each pair $\{ R_p, R_q \}$; and since $R_{i(s_j)}$ is contained in $H^+(s_j)$, then there are m additional edges in G, one for each side s_j. The total number of edges in G is at least $\binom{n}{2} + m$.

The graph G is bipartite, then there is a vertex v_t with degree at least $\left[\binom{n}{2} + m \right] / m = \left[\binom{n}{2} / m \right] + 1 \geq \left[\binom{n}{2} / (9n-12) \right] + 1$

which is greater than $(3n+55)/54$. This means that the closed halfplane $H^+(s_t)$ contains a subcollection $R(s_t)$ of R with at least $\lceil(3n+55)/54\rceil$ polygons. The corresponding subcollection $P(s_t)$ of P is well supported since $P_{i(s_t)} \in P(s_t)$ and the side of $P_{i(s_t)}$ which is contained in s_t lies in the boundary of the convex closure of $P(s_t)$. ∎

Theorem 3.- Let $P = \{P_1, P_2, \ldots, P_n\}$ be a collection of $n \geq 2$ plane convex polygonal regions with pairwise disjoint relative interiors. There is a side s of a region P_i such that the line that supports s separates P_i from at least $\lceil(3n+1)/54\rceil$ sets in P.

Proof.- Let $R = \{R_1, R_2, \ldots, R_n\}$ be as in Lemma 1 and $\{l_1, l_2, \ldots, l_m\}$ be the set of lines supporting the sides of $R_1, R_2, \ldots, R_n$; by Lemma 1, $m \leq 9n-12$. If a line l supports a side of c sets in R, we include c copies of l in L. Therefore we may associate to each l_k a unique set $R(l_k)$ in R such that l_k contains a side of $R(l_k)$. For $i = 1, 2, \ldots, m$, let $H^-(l_k)$ be the closed halfplane determined by l_k that does not include $R(l_k)$.

Define a bipartite graph F with a vertex u_i for each set R_i and a vertex v_k for each line l_k. The graph F has an edge $u_i v_k$ if the set R_i is contained in $H^-(l_k)$.

For every pair of polygonal regions $\{R_i, R_j\}$, there is at least one side s of one of them such that the line supporting s separates R_i from R_j. Therefore F has at least one edge for each pair $\{i, j\}$ with $1 \leq i < j \leq n$. Since F is bipartite, there is a vertex v_k whose degree in F is at least $\binom{n}{2} / m \geq \binom{n}{2} / (9n-12) > (3n+1)/54$.

The closed halfplane $H^-(l_k)$ contains at least $\lceil(3n+1)/54\rceil$ sets in R. Since $R(l_k)$ is not contained in $H^-(l_k)$ then l_k separates $R(l_k)$ from at least $\lceil(3n+1)/54\rceil$ sets in R. To end the proof, let s be such that $R(l_k) = R_s$. The line l_k separates P_s from at least $\lceil(3n+1)/54\rceil$ sets in P and by Property 2 in Lemma 1, l_k supports a side of P_s. ∎

Corollary 4.- In any collection F of $n \geq 2$ plane convex sets with pairwise disjoint relative interiors, there is a pair of sets A and B, such that any line that separates A from B separates either A or B from at least $\lceil(3n+1)/54\rceil$ sets in F.

Proof.- Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a collection of convex sets in the plane with pairwise disjoint relative interiors. Let $L^0 = \{ l^0_{ij} : 1 \le i < j \le n \}$ be a set of lines such that l^0_{ij} separates A_i from A_j.

Each line l^0_{ij} determines two closed halfplanes H^0_{ij} and H^0_{ji} containing A_i and A_j respectively. For $i = 1, 2, \ldots, n$ let $R_i = \bigcap H^0_{ik}$, $1 \le k \le n$, $k \ne i$. Clearly $R^0 = \{ R^0_1, R^0_2, \ldots, R^0_n \}$ is a collection of convex plane polygonal regions with pairwise disjoint relative interiors. By Theorem 3, there is a side s of a region $R^0_{i(0)}$ such that the line $l^0_{i(0)j(0)}$ that supports s separates $A_{i(0)}$ from at least $\lceil (3n+1)/54 \rceil$ sets in F. If there is another line l^1 that separates $A_{i(0)}$ from $A_{j(0)}$ but separates neither $A_{i(0)}$ nor $A_{j(0)}$ from at least $\lceil (3n+1)/54 \rceil$ sets in F, let $L^1 = \{ l^1_{ij} : l^1_{i(0)j(0)} = l^1$, and $l^1_{ij} = l^0_{ij}$ if $\{i,j\} \ne \{i(0),j(0)\} \}$. For $1 \le i < j \le n$, let H^1_{ij} and $R^1_i = \bigcap H^0_{ik}$, $1 \le k \le n$, $k \ne i$ be defined as above and apply Theorem 3 to $R^1 = \{ R^1_1, R^1_2, \ldots, R^1_n \}$. The process is repeated until a stage t is reached, where the sets $A_{i(t)}$ and $A_{j(t)}$ are such that every line separating them, separates either $A_{i(t)}$ or $A_{j(t)}$ from at least $\lceil (3n+1)/54 \rceil$ sets in F. Notice that $t \le \binom{n}{2}$ since $\{i(q),j(q)\} \ne \{i(p),j(p)\}$ whenever $q \ne p$. ■

We believe that the bounds given in Theorems 2 and 3 are not optimal; we think that the right values are close to $n/3$.

4 References.-

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