Every Arrangement Extends to a Spread

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An *arrangement of pseudolines* in the Euclidean plane $\mathbf{E}^2$ is a finite family of simple curves in $\mathbf{E}^2$ such that every two curves intersect at precisely one point, at which they cross. A *spread of pseudolines* in $\mathbf{E}^2$ is an infinite family of simple closed curves in $\mathbf{E}^2$ such that:

1. every two curves intersect at precisely one point, at which they cross;
2. there is a bijection L from the unit circle C to the family of curves such that $L(p)$ is a continuous function (in the Hausdorff metric) of $p \in C$.

We prove the following conjecture of Grünbaum [1]:

Theorem 1 *Every arrangement of pseudolines in $\mathbf{E}^2$ may be embedded in a spread of pseudolines.*

Using a stereographic projection, the Euclidean plane can easily be mapped to the interior of a disk, with pseudolines in $\mathbf{E}^2$ mapping to curves on the disk with endpoints on the circle bounding the disk. Stein [2] proved that an arrangement of pseudolines in a disk is combinatorially equivalent to some arrangement of pseudolines in a regular $2n$-gon such that each face in the arrangement is a convex polygon. The pseudolines have antipodal vertices on the $2n$-gon as endpoints. (Two points, $p, \bar{p}$, on the boundary of the regular $2n$-gon are *antipodal* if the line through $p, \bar{p}$ passes through the center of the polygon.) To prove Theorem 1 we need only show that a finite family of pseudolines on the $2n$-gon can be extended to an infinite family where every point p on the boundary of the $2n$-gon lies on exactly one pseudoline $L(p)$ and $L(p)$ is a continuous function of p.

Let l and l' be two curves on the $2n$-gon P with distinct antipodal endpoints $p, \bar{p}$ and $p', \bar{p}'$, respectively. Let q be some point of intersection of l and l' at which they cross. q divides l into two segments s with endpoints p, q and $\bar{s}$ with endpoints $\bar{p}, q$. Similarly, q divides l' into two segments s' and $\bar{s}'$ with endpoints p', q and $\bar{p}', q$, respectively. We say that the q is a *proper intersection point* of l and l' if l and l' cross at q and $s, s', \bar{s}, \bar{s}'$ occur in clockwise order around q if and only if $p, p', \bar{p}, \bar{p}'$ occur in clockwise order around P.

We can replace the global condition that curves intersect at precisely one point, at which they cross, by the local condition that every point of intersection is proper.

Lemma 2 *Two curves with antipodal endpoints on a $2n$-gon intersect at precisely one point, at which they cross, if and only if every point of intersection of the two curves is a proper intersection point.*

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Now, we show how to extend an arrangement to a continous family of curves such that every intersection is proper. Let $\mathcal{L}$ be an arrangement of n pseudolines in a $2n$-gon P with endpoints on opposing vertices. The pseudolines in $\mathcal{L}$ partition P into a 2-dimensional cell complex, consisting of a set of faces $F(\mathcal{L})$, edges $E(\mathcal{L})$, and vertices $V(\mathcal{L})$. We assume that the faces and edges do not contain their boundary points. Edges of P are considered edges in $E(\mathcal{L})$ and their endpoints are vertices in $V(\mathcal{L})$.

Let $\mathcal{A}$ be the set of edges of P. For each edge $a \in \mathcal{A}$, there are two pseudolines, $l_a, l'_a \in \mathcal{L}$, whose endpoints match the endpoints of a. Let $\gamma_a = l_a \cap l'_a$. Each edge $a \in \mathcal{A}$ also has an opposing edge $\bar{a} \in \mathcal{A}$, where $\{l_a, l'_a\} = \{l_{\bar{a}}, l'_{\bar{a}}\}$.

For each $a \in \mathcal{A}$, we can define an ordering relation $R_a(\mathcal{L})$ on the faces, edges and vertices of an arrangement $\mathcal{L}$ as follows. Assume edge e and vertex v are on the boundary of face f. Let $e <_a f$ if the pseudoline $l \in \mathcal{L}$ containing e strictly separates a from f. (We use $<$ instead of $<_a$ whenever the subscript is clear from the context.) Otherwise, let $e > f$. Let $v < f$ if all the pseudolines $l \in \mathcal{L}$ containing v strictly separate a from f. Let $v > f$ if all the pseudolines $l \in \mathcal{L}$ containing v strictly separate $\bar{a}$ from f. $e < f$ and $e > f'$ for exactly one f and one f'. Similarly, $v < f$ and $v > f'$ for exactly one f and one f'. It is easy to see that

Lemma 3 *$R_a(\mathcal{L})$ is a partial ordering on the faces, edges and vertices of $\mathcal{L}$.*

Let $R^*_a(\mathcal{L})$ be the transitive closure of $R_a(\mathcal{L})$. Let G_a be the set of all faces, edges and vertices which have some relation in $R^*_a(\mathcal{L})$ to γ_a.

$$G_a = \{g \in F(\mathcal{L}) \cup E(\mathcal{L}) \cup V(\mathcal{L}) : g < \gamma_a \text{ or } g > \gamma_a \text{ or } g = \gamma_a \text{ in } R^*_a(\mathcal{L})\}.$$

The lines l_a and l'_a partition P into four regions. G_a consists of all the faces, edges and vertices lying in the regions containing a and $\bar{a}$.

For each $f \in F$, let $B(f)$ be the set of points on the boundary of f. $B(f) - l_a - l'_a$ denotes the set of points on the boundary of f which do not lie on line l_a or line l'_a. For each $a \in \mathcal{A}$ and $f \in F_a$, define:

$$B_a(f) = \{p \in B(f) : p \in g \in G_a, g <_a f\}.$$

Note that if f is the face with a on its boundary, then $B_a(f) = a$. (The closure of a set of points S is denoted $cl(S)$.)

Let $B^-_a(f)$ and $B^+_a(f)$ be the endpoints of $B_a(f)$ with $B^-_a(f)$, $B_a(f)$ and $B^+_a(f)$ occurring in clockwise order around f.

If p lies on some edge or vertex $g \in G_a$, then there is a unique f such that $g <_a f$. Thus there is a unique face f such that $p \in B_a(f)$. Let f^*_a be the unique face where $\gamma_a \in B_a(f^*_a)$. Note that $B_a(f^*_a) = \gamma_a$.

The following three lemmas are simple observations.

Lemma 4 *For all faces $f \in G_a - f^*_a$, the set of points $B_a(f)$ is an open, connected curve.*

Lemma 5 *For every face $f \in G_a$, $B_a(f) \neq \emptyset$.*

Lemma 6 *If $a, b, \bar{a}, \bar{b} \in \mathcal{A}$ occur in clockwise order around P, then $B^-_a(f)$, $B^-_b(f)$, $B^+_{\bar{a}}(f)$, $B^+_{\bar{b}}(f)$ (not necessarily distinct) occur in clockwise order f.*

The next lemma is the crucial ingredient in our construction. It provides us with a construction of the spread locally in each face so that the required global properties will be satisfied.

Lemma 7 *There exist a set of functions $\{\psi_a : a \in \mathcal{A}\}$, where ψ_a maps $B_a(f)$ to $B_{\bar{a}}(f)$ for each face $f \in G_a - f^*_a - f^*_{\bar{a}}$ such that:*

1. *ψ_a is continuous, one-to-one and onto;*
2. *$\psi_{\bar{a}}$ is the inverse of ψ_a;*
3. *for every distinct $p, p' \in B_a(f)$, the line segment from p to $\psi_a(p)$ does not cross the line segment from p' to $\psi_a(p')$;*

4. if $p \in B_a(f) \cap B_b(f)$ and $a, b, \bar{a}, \bar{b}$ occur in clockwise order around P, then p, $\psi_a(f)$, $\psi_b(f)$ are distinct points occurring in clockwise order around f.

Proof: We prove the lemma by constructing a family of functions $\{\psi_a : a \in \mathcal{A}\}$ with the desired properties. Assume we have defined ψ_a for all $a \in A' \subseteq A$, where $a \in A'$ implies $\bar{a} \in A'$. We will show how to define ψ_a and $\psi_{\bar{a}}$ for some $a, \bar{a} \in A - \{A'\}$.

Choose any face $f \in G_a - f_a^* - f_{\bar{a}}^*$. Let $A'' = \{a' \in A' : f \in G_{a'}\}$. Sort the edges in $A'' \cup \{a\}$ in clockwise order around P. Let a_- and a_+ be the two edges of A'' which immediately proceed and immediately follow a in clockwise order around P. If $A'' = \emptyset$, then a_- and a_+ are undefined.

Order the points in $cl(B_a(f))$ clockwise around f. Thus, $p < p'$, $p, p' \in cl(B_a(f))$, if travelling clockwise on $cl(B_a(f))$ one first encounters p and then p'. Similarly, order the points in $cl(B_{\bar{a}}(f))$ clockwise around f.

Let $q^- = B_{\bar{a}}^-(f)$ and $q^+ = B_{\bar{a}}^+(f)$. By Lemma 6, $q^- \in cl(B_{\bar{a}_-}(f))$ and $q^+ \in cl(B_{\bar{a}_+}(f))$. Thus:

$$p^- = \lim_{q \to q^-} \psi_{a_-}^{-1}(q), \quad \text{and}$$
$$p^+ = \lim_{q \to q^+} \psi_{a_+}^{-1}(q)$$

is well-defined.

Assume $p^- \in B_a(f)$. Again by Lemma 6, if $p > p^-$, $p \in B_a(f)$, then $p \in B_{a_-}(f)$. Define a function μ^- from $B_a(f)$ to $B_{\bar{a}}(f)$ where $\mu^-(p) = \psi_{a^-}(p)$ for all $p \geq p^-$ and $\mu^-(p) = q^-$ for all $p \leq p^-$. If $p^- \notin B_a(f)$, then let $\mu^-(p) = q^-$ for all $p \in B_a(f)$.

Similarly, if $p^+ \in B_a(f)$, then $\mu^+(p) = \psi_{a_-}(p)$ for all $p \leq p^+$ and $\mu^+(p) = q^+$ for all $p \geq p^+$. Otherwise $\mu^+(p) = q^+$ for all $p \in B_a(f)$.

Clearly μ^- and μ^+ are continous functions of $p \in B_a(f)$. We also claim that $\mu^-(p) < \mu^+(p)$ for all $p \in B_a(f)$ and that μ^- and μ^+ are monotonically decreasing functions of p, i.e., if $p < p'$, then $\mu^-(p) \geq \mu^-(p')$ and $\mu^+(p) \geq \mu^+(p)$. First note that by Lemma 6, $q^+ \notin B_{\bar{a}_-}(f)$ and $q^- \notin B_{\bar{a}_+}(f)$. Thus for all $p \in B_a(f)$, $\mu^-(p) \neq q^+$ and $\mu^+(p) \neq q^-$. It follows that if $\mu^-(p) = q^-$ or $\mu^+(p) = q^+$, then $\mu^-(p) < \mu^+(p)$.

Assume $\mu^-(p) = \psi_{a_-}(p) \neq q^-$ and $\mu^+(p) = \psi_{a_+}(p) \neq q^+$. By property 4 above, p, $\psi_{a_-}(p)$ and $\psi_{a_+}(p)$ appear in clockwise order, so $\psi_{a_-}(p) < \psi_{a_+}(p)$.

By property 3 above, if $p < p'$, $p, p' \in B_a(f) \cap B_{a_-}(f)$, $\psi_{a_-}(p), \psi_{a_-}(p') \in B_{\bar{a}}(f) \cap B_{\bar{a}_-}(f)$, then $\psi_{a_-}(p) > \psi_{a_-}(p')$. Thus $\mu^-(p)$ is a monotonically decreasing function of p. Similarly, $\mu^+(p)$ is a monotonically increasing function of p.

We now choose any continuous monotonically decreasing function of p lying between μ^- and μ^+ to be ψ_a. Since ψ_a is monotonically decreasing, the line segment from p to $\psi_a(p)$ does not cross the line segment from p' to $\psi_a(p')$ for any $p, p' \in B_a(f)$.

Assume $p \in B_a(f) \cap B_b(f)$ and $a, b, \bar{a}, \bar{b}$ occur in clockwise order around P. By the choice of a_+, $a, a_+, b, \bar{a}, \bar{a}_+, \bar{b}$ occur in clockwise order around P. By property 4, p, $\psi_{a_+}(f)$, $\psi_b(f)$ occur clockwise around P. By construction of ψ_a, p, $\psi_a(f)$, $\psi_{a_+}(f)$ occur clockwise around P. Thus p, $\psi_a(f)$, $\psi_{a_+}(f)$, $\psi_b(f)$ occur in clockwise order around p, showing property 4 holds. If $b, a, \bar{b}, \bar{a}$ occur in clockwise order around P, then using a_- one can again show property 4 holds.

We repeat the above procedure for each face, defining $\psi_a(p)$ for all $p \in B_a(f)$, $f \in G_a - f_a^* - f_{\bar{a}}^*$. We then let $\psi_{\bar{a}}(p) = \psi_a^{-1}(p)$. $\psi_{\bar{a}}(p)$ is also a continuous monotonically decreasing function of $p \in B_{\bar{a}}(f)$. Showing it has all the properties above is a simple exercise. □

Extend each functions ψ_a to $B_a(f_{\bar{a}}^*)$, by letting $\psi_a(p) = \gamma_a$, $p \in B_a(f_{\bar{a}}^*)$. Let $\psi_a^i(p)$ be ψ_a applied i times to p. Note that $\psi_a^0(p) = p$.

Let $(p, \psi_a(p), \psi_a^2(p), \ldots, \psi^k(p) = \gamma_a)$ be the polygonal curve consisting of line segments $(\psi_a^i(p), \psi_a^{i+1}(p))$, $0 \leq i < k$. For each $p \in a \in \mathcal{A}$ there is an antipodal point $\bar{p} \in \bar{a} \in \mathcal{A}$. Let $L(p)$ be the union of the polygonal curves $(p, \psi_a(p), \psi_a^2(p), \ldots, \psi_{\bar{a}}^k(p) = \gamma_a)$ and $(\bar{p}, \psi_a(\bar{p}), \psi_{\bar{a}}^2(\bar{p}), \ldots, \psi_{\bar{a}}^{k'}(\bar{p}) = \gamma_a)$.

Proof of Theorem 1: $\mathcal{S}\{L(p) : p \in a \in \mathcal{A}\} \cup \mathcal{L}$ is a spread of pseudolines containing $\mathcal{L}$. Let l and l' be two pseudolines in $\mathcal{S}$. If $l \in \mathcal{L}$ and $l' \in \mathcal{L}$, then, by definition, they intersect in exactly one point.

Assume $l' \in \mathcal{L}$ but $l \notin \mathcal{L}$. Since the endpoints of l and l' are antipodal, l and l' must intersect in at least one point $q^* \in g^* \in G_a$. If there is a line segment in l' connecting $g \subseteq B_a(f)$ to $g' \subseteq B_{\bar{a}}(f)$, then $g <_a f <_a g'$. If $g \in G_a$ intersects l' between q^* and a, then $g < g^*$ in R^*_a. If $g \in G_a$ intersects l' between q^* and $\bar{a}$, then $g > g^*$ in R^*_a. Thus g cannot be a vertex or edge lying on l, and so l intersects l' only at q^*.

Assume $l \notin \mathcal{L}$ and $l' \notin \mathcal{L}$. If l and l' have endpoints in the same arc $a \in \mathcal{A}$, then l and l' intersect at γ_a where they cross. Let l and l' have endpoints in different arcs, a and a', respectively, where $a, a', \bar{a}, \bar{a}'$ occur in clockwise order around P. We show that every point of intersection q^* of l and l' is proper. We consider three cases, depending upon whether q^* lies on a face, an edge or a vertex in the cell complex generated by $\mathcal{L}$.

First, assume q^* lies in the interior of some face $f \in G_a \cap G_{a'}$. Let $q = B_a(f) \cap l$, $\bar{q} \in B_{\bar{a}}(f) \cap l$, $q' = B_{a'} \cap l'$, $\bar{q}' = B_{\bar{a}'} \cap l'$. We need to show that $q, q', \bar{q}, \bar{q}'$ occur in clockwise order around f. It suffices to show that any three of these points lie in proper order around f; the fourth point is antipodal to one of the three and automatically falls into the proper position.

If $q \notin B_{a'}(f) \cup B_{\bar{a}'}(f)$, then by Lemma 6, $\bar{q}', q, q'$ must lie in clockwise order, proving q^* is a proper intersection point. If $q \in B_{a'}(f)$. then by Lemma 7, property 4, $q, \bar{q}, \psi_{a'}(q)$ lie in clockwise order. Since the line segment $q', \bar{q}'$ does not intersect $q, \psi_{a'}(q)$, points $q, q', \bar{q}$ must lie in clockwise order, again proving q^* is a proper intersection point. A similar argument holds if $q \in B_{\bar{a}'}(f)$.

Next, assume q^* lies on some edge $e \in G_a \cap G_{a'}$. e lies on the boundary of two faces, f and f'. Without loss of generality, assume $q^* \in B_a(f) \cap B_{a'}(f)$ and $q^* \in B_{\bar{a}}(f') \cap B_{\bar{a}'}(f')$. By Lemma 7, property 4, $\psi_a(q^*)$, $\psi_{a'}(q^*)$, q^* occur in clockwise order around f and $\psi_{\bar{a}}(q^*)$, $\psi_{\bar{a}'}(q^*)$, q^* occur in clockwise order around f'. It follows that q^* is a proper intersection point.

Finally, assume q^* lies on some vertex $v \in G_a \cap G_{a'}$. Let l'' be some line in $\mathcal{L}$ containing v. Without loss of generality, assume l'' separates a and a' from $\bar{a}$ and $\bar{a}'$. Assume some other line $l''' \in \mathcal{L}$ passing through q^* separates a from a'. As argued above, l' and l'' intersect l'' and l''' in exactly one point. Thus, l' must intersect l'' only at q^* and q^* is a proper intersection point.

If no other line from $\mathcal{L}$ passing through q^* separates a from a', then $q^* \in B_a(f) \cap B_{a'}(f)$ and $q^* \in B_{\bar{a}}(f') \cap B_{\bar{a}'}(f')$. The argument is then the same as the case where q^* lies on some edge $e \in G_a$.

We have shown that every point of intersection of l and l' is proper. By Lemma 2, l and l' intersect precisely once. If follows that any two curves in $\mathcal{S}$ intersect precisely once, where they cross. For each point p on the boundary of P, there is a unique pseudoline $L(p)$ with endpoint p. Since ψ_a is a continuously varying function of $p \in a$, $L(p)$ is a continuously varying function of $p \in a$. If p' is an endpoint of a, then $L(p') \in \mathcal{L}$ is the limit of $L(p)$ as p approaches p'. Thus $\mathcal{S}$ is a spread of pseudolines containing $\mathcal{L}$. □

References

[1] GRÜNBAUM, B. *Arrangements and Spreads.* American Mathematical Society, Providence, Rhode Island, 1972.

[2] STEIN, S. Convex maps. *Proc. Amer. Math. Soc. 2* (1951), 464–466.