Bicriteria Shortest Path Problems in the Plane

(extended abstract)

Esther M. Arkin [*] Joseph S. B. Mitchell [†] Christine D. Piatko [‡]

1 Introduction

There have been many algorithms in computational geometry that produce optimal paths according to some notion of "shortest". The problem of finding shortest (Euclidean or L_1 length) paths among obstacles in the plane is well-studied [1,8], and there have been recent works also on the problem of finding shortest paths according to other notions of "length": link distance [10,11], weighted length [9], and minimum-time [3].

In this paper we study various *bicriteria* path problems in a geometric setting. We consider several pairs of criteria for planar paths, including: total turn and path length, path length measured according to two different norms (L_p and L_q), and path length within two or more classes of regions. As is the case for the general bicriteria path problem on graphs, many of these problems are NP-complete. In addition to proving these hardness results, we give pseudo-polynomial time algorithms for some cases.

In a closely related paper, [2], we present a polynomial-time approximation algorithm for computing bicriteria paths within a simple polygon, according to the two criteria of Euclidean length and link distance. We compute (approximately) the shortest path from s to t that uses only k links.

2 Review of Bicriteria Paths in Graphs

First, we review the general result for bicriteria paths in graphs [4, p. 214], which forms the basis for many of our constructions.

Theorem 1 *In a graph (V, E) with positive integer weights w_i and positive integer lengths l_i on its edges, and two distinguished nodes s and t, the problem "Does there exist a path from s to t with weight $\leq W$ and length $\leq L$?" is NP-complete.*

Proof. We use a reduction from Partition. In the Partition problem we are given a set N of items with positive integer weights a_i, and ask "Does there exist a subset $S \subset N$ such that $\sum_{i \in S} a_i = \frac{1}{2}\sum_{i=1}^{n} a_i$?" Consider a graph with $n+1$ nodes, with $s = v_1$ and $t = v_{n+1}$. Draw two edges joining node v_i to v_{i+1}, a "top" edge with length 0 and weight a_i, and a "bottom" edge with length a_i and weight 0. Set $L = W = \frac{1}{2}\sum_{i=1}^{n} a_i$. A path of length $\leq L$ and weight $\leq W$ on this graph yields a partition into top and bottom edges that solves the Partition problem. □

The above proof is not affected if we add a constant C to the lengths (weights) of the top and bottom edges joining v_i and v_{i+1}, while adding C to L (W). What is important is that the *difference* between the two lengths (weights) equals a_i.

Partition is a weakly NP-complete problem, so it is not surprising that there are pseudo-polynomial time algorithms for this bicriteria problem. In fact, the Bellman-Ford dynamic programming method for shortest paths [7, p. 74] provides a polynomial time algorithm for the equal-length (or equal-weight) version. If the lengths or the weights of the edges are bounded, we can solve the problem in polynomial time, by breaking the arcs into "unit" length or weight segments. This implies a pseudo-polynomial time algorithm for the general bicriteria problem on graphs.

For correctness we must note that the constructions presented in this paper sometimes require irrational coordinates. Since irrational coordinates cannot be generated in polynomial time, our reductions, if done precisely, are not polynomial. However, since there are rational points arbitrarily close to irrational points, we can choose rational coordinates in polynomial time such that the chosen points will differ by at most ϵ from the desired ones. Usually we will be connecting n gadgets. We can choose an ϵ such that $n\epsilon$ is small enough so that the sum of small differences in length over all the gadgets will not affect the optimal solution. For brevity in this abstract, we will not indicate all the perturbations that are necessary.

3 Total Turn and Length

One version of the geometric bicriteria path problem in the plane asks us to find a path from s to t that minimizes the length and total turn of the path. (The *total turn* of a path is the sum of the absolute values of changes in θ over the path.)

Any pareto-optimal path for total turn and length

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[†] SORIE, Cornell University, Ithaca, NY 14853. Partially supported by NSF grant ECSE-8857642, and by a grant from Hughes Research Laboratories.

[‡] Dept. of Computer Science, Cornell University, Ithaca, NY 14853. Partially supported by NSF Grant ECSE-8857642, a grant from Hughes Research Laboratories and a National Science Foundation Graduate Fellowship.

must lie on the visibility graph. (A *pareto-optimal* path is one that is not improvable in one of the two criteria without increasing the other criteria.) If not, we can shortcut along a chord of the path, improving both the length and the total turn. A corollary of this is that in a simple polygon the (unique) shortest path is the only pareto-optimal path. For polygons with holes however, we have the following result:

Theorem 2 *The problem "Does there exist a path from s to t, in a polygon with holes, whose length is* $\leq L$ *and whose total turn is* $\leq \theta$*?" is NP-complete.*

Proof. The proof is based on the graph construction used above for bicriteria paths in graphs, with added constants so there are no zero lengths or weights to the edges. We again use a reduction from Partition, scaled so that each a_i is less than π (and thus may not be integer). We construct a planar graph with $n+1$ nodes, with $s = v_1$ and $t = v_{n+1}$. We draw two "edges" from each v_i to v_{i+1}, one with length $\lambda + a_i$ and total turn 2π and one with length λ and total turn $2\pi + a_i$, where λ is a constant. The claim is that the first edge can be drawn in the plane with three bends, with length $\lambda + a_i$ and turn 2π, and the second edge can be drawn with 3 edges, with total length λ and total turn $2\pi + a_i$ (see Figure 1). We draw a corridor of constant length K (where K is bigger than any $\lambda + a_i$) so that consecutive gadgets will not overlap.

Our obstacles will be the complement of the edges we have drawn. Thus, a partition will exist if and only there exists a path with total turn $\leq 2\pi n + \frac{1}{2}\sum_{i=1}^{n} a_i$ and length $\leq n\lambda + \frac{1}{2}\sum_{i=1}^{n} a_i$. □

However, the problem is not strongly NP-complete:

Theorem 3 *There exists a pseudo-polynomial time algorithm for the problem of minimizing total turn and length.*

Proof. We know the optimal path must lie on the visibility graph, so we can map visibility graph edges to a graph G. Each visibility graph edge e between u and v will be split into two directed edges. The directed edge from u to v is changed into an edge between the nodes u_{e_out} and v_{e_in}. Similarly, the edge from v to u becomes an edge between v_{e_out} and u_{e_in}. Both edges are given length $||u, v||$ and weight (corresponding to turn) zero.

Assume the visibility graph (VG) edges are ordered around the vertex v. Assume that the extensions of the visibility graph edges are also ordered around v. For each VG edge e directed into v we find the VG edge f directed out of v that is clockwise to e's extension. We connect v_{e_in} to v_{f_out}, and give the new edge a weight θ, where θ is the amount of turn from e to f, and length 0. Similarly, we connect v_{e_in} to v_{g_out} where g is the VG edge counterclockwise to e's extension.

If the n obstacle vertices have integer coordinates (of maximum size N), the smallest angle formed by any pair of VG edges is $\theta_{min} = \frac{c}{N^2}$, for a constant c. We define $\Delta\theta = \frac{\theta_{min}}{2n}$, and round all weights θ to integer multiples of $\Delta\theta$. Since no paths of interest have more than n turns, we can argue that measuring angles to within the resolution $\Delta\theta$ is sufficient for solving the bicriteria path problem. By replacing an edge whose weight is $k \cdot \Delta\theta$ by k edges each with unit weight, and applying the dynamic programming algorithm of Bellman-Ford on the graph G, we can find an optimal solution within time $O((nN^2E)^3)$.

4 Minimizing Both L_p and L_q Length

Suppose we would like to minimize the L_p and L_q lengths of a path from s to t simultaneously. We can show that this problem is also NP-hard. Here we give a proof for the L_1 and L_2 norms. This proof can be generalized to any two L_p and L_q norms (where $p \neq q$). It can also be generalized to two convex distance functions that are not similar under scaling.

Theorem 4 *The problem "Does there exist a path from s to t whose* L_1 *length is* $\leq A$ *and whose* L_2 *length is* $\leq B$*?" is NP-complete.*

Proof. (Sketch.) We use a reduction from Partition, similar to the one for the bicriteria path problem in graphs. First, we make a gadget that corresponds to the nodes v_i and v_{i+1} and the 2 edges between them (Figure 2). We start with an isosceles right triangle with base b_i, height b_i and hypotenuse c_i. We add a skinny vertical "hump" of total length x_i to the hypotenuse. (Note that the lengths we refer to here will be off by a small amount. By adding the vertical "hump" we take a small amount away from the length of c_i, and the hump cannot be perfectly vertical. However, such differences can be made small enough so that they do not affect the structure of the proof.) The exact values of b_i and x_i will be chosen later. The L_2 length of c_i is $\sqrt{2}\,b_i$, and the L_1 length of c_i is $2b_i$. There will be only 2 paths from v_i to v_{i+1}. The upper path, following the hypotenuse and the hump, has L_2 length $x_i + \sqrt{2}\,b_i$, and L_1 length $x_i + 2b_i$. The lower path has L_2 and L_1 length $2b_i$. We choose x_i so that the upper path is longer than the lower path by a_i (the value of the ith item in the Partition problem) in the L_1 norm and shorter by a_i in the L_2 norm. We want $x_i = (2b_i + a_i) - 2b_i$, that is, $x_i = a_i$. We also want $\sqrt{2}\,b_i + a_i = 2b_i - a_i$, which implies we should choose $b_i = (2 + \sqrt{2})\,a_i$. We connect n of these gadgets along a diagonal line, and take as obstacles the complements of the paths drawn. □

5 Travel Through Multiple Regions

Suppose the plane is partitioned into red and blue regions. We can ask for the path from s to t that minimizes travel in both the red and blue regions. For any L_p metric this problem is NP-hard. To prove this we first show a special version of the Knapsack problem is NP-complete. The reduction for multiple regions will mimic this proof.

In Fractional Knapsack we are given a set N of items, each with a value v_i and weight w_i, and bounds V and W. A solution to Fractional Knapsack, will be a set $S \subset N$ of whole items and a set $F \subset N$ of fractional items, with $S \cap F = \emptyset$, such that the knapsack has value $\geq V$ and weight $\leq W$. Let f_i be the fraction of item i taken, i.e. $0 \leq f_i \leq 1$. The value of a knapsack is the sum of the value of whole items taken, plus the fractional *weight* of fractional items taken, i.e. $\sum_{i \in S} v_i + \sum_{i \in F} f_i \cdot w_i$. Alternatively, we can think of the value of a knapsack as the value of items completely taken plus any remaining capacity, i.e., $\sum_{i \in S} v_i + (W - \sum_{i \in S} w_i)$. The weight of the knapsack is just the weight of whole items plus the appropriate fraction of the weight of fractional items, i.e. $\sum_{i \in S} w_i + \sum_{i \in F} f_i \cdot w_i$.

Theorem 5 *The Fractional Knapsack problem, "Do there exist $S \subset N$ and $F \subset N$ such that the value of $S \cup F$ is $\geq V$ and the weight of $S \cup F \leq W$?" is NP-complete.*

Proof. We use a reduction from Partition. Let $W = \frac{1}{2} \sum_{i \in N} a_i$. For item i, let $w_i = a_i$ and let $v_i = 2(W+1) \cdot a_i$. Let $V = 2(W+1) \cdot \frac{1}{2} \sum_{i \in N} a_i$. Suppose we are given a solution to Fractional Knapsack. We know the weight of the knapsack is $\leq W$, i.e. $\sum_{i \in S} a_i + \sum_{i \in F} f_i \cdot a_i \leq W = \frac{1}{2} \sum_{i \in N} a_i$. The value of the knapsack is $\geq V$, i.e.

$$\sum_{i \in S} 2(W+1) \cdot a_i + \sum_{i \in F} f_i \cdot a_i \geq (W+1) \sum_{i \in N} a_i = V.$$

Since $\sum_{i \in F} f_i \cdot a_i \leq W$ we can subtract $\sum_{i \in F} f_i \cdot a_i$ from the left and W from the right to get

$$2(W+1) \sum_{i \in S} a_i \geq (W+1) \sum_{i \in N} a_i - W$$

$$\Longrightarrow \sum_{i \in S} a_i \geq \frac{1}{2} \sum_{i \in N} a_i - \frac{W}{2(W+1)}.$$

We know that $\frac{W}{2(W+1)}$ is a fraction, and in particular it is less than 1/2. The sum on left hand side of the equation is an integer. The sum on the right hand side of the equation is either an integer or an integer plus 1/2. In either case, since the fraction $\frac{W}{2(W+1)}$ is small enough, we know that $\sum_{i \in S} a_i \geq \frac{1}{2} \sum_{i \in N} a_i$. We already know the weight of the knapsack implies that $\sum_{i \in S} a_i \leq \frac{1}{2} \sum_{i \in N} a_i$. Thus, the Fractional Knapsack solution solves Partition. □

We can now use a similar proof technique to show that minimizing travel through two regions simultaneously is NP-complete.

Theorem 6 *The problem "Does there exist a path from s to t whose L_1 length in red is $\leq R$ and whose L_1 length in blue is $\leq B$?" is NP-complete.*

Proof. (Sketch.) We use a reduction from Partition, based on the Fractional Knapsack reduction, where the v_i's, w_i's, V and W are chosen the same way as above (note that the v_i's are larger than all w_i's). We create "tunnels" between s and t such that the length of travel in the blue region for the ith item is at least $c - w_i$ (where c is chosen so $c - w_i > W$ for all i). We then create a red barrier such that travelling in red would cost w_i, corresponding to choosing the item, but going around the red barrier through a blue tunnel would cost $(v_i - w_i)/2 + w_i + (v_i - w_i)/2$, i.e. v_i, corresponding to leaving the item (see Figure 3).

We can then think of the value of choosing an item as the amount of "savings" if we shortcut through the red region instead of travelling through blue. For now, assume that if an item is not chosen, the entire blue path is followed. Thus, the length of the path in blue is $c \cdot |N| - \sum_{i \in N} w_i + \sum_{i \notin S} v_i$. We choose $R = W$ and $B = c \cdot |N| - \sum_{i \in N} w_i + \sum_{i \in N} v_i - V$. Thus, if the length in blue is $\leq B$, $\sum_{i \notin S} v_i \leq \sum_{i \in N} v_i - V \Longrightarrow \sum_{i \in S} v_i \geq V$.

However, the path can "cut corners" through the red region, i.e. it can cut corners of the path to allow a little red to be chosen. This corresponds to choosing a fractional item. If an item is partially selected its value will be its length in red (which corresponds to its weight in the Fractional Knapsack problem). By connecting n of these blue tunnels together, the same proof technique used for Fractional Knapsack can be used to prove our problem NP-complete. □

We can modify this proof to work for any L_p metric, by using a variation of Fractional Knapsack in which fractional items contribute an appropriate constant times their fractional weight as value to the knapsack.

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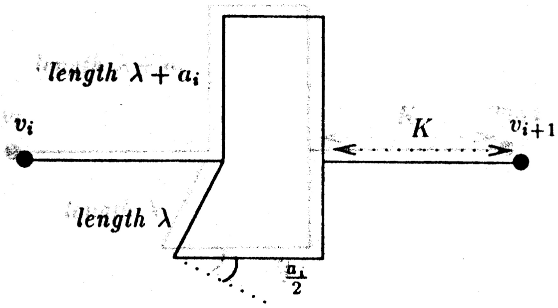

Figure 1: Gadget for total turn and length.

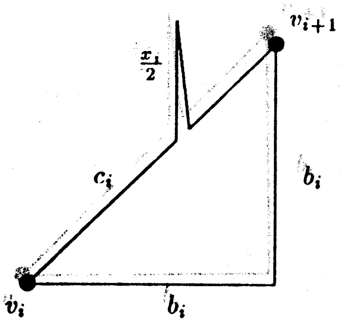

Figure 2: Corridors for L_1 and L_2 length.

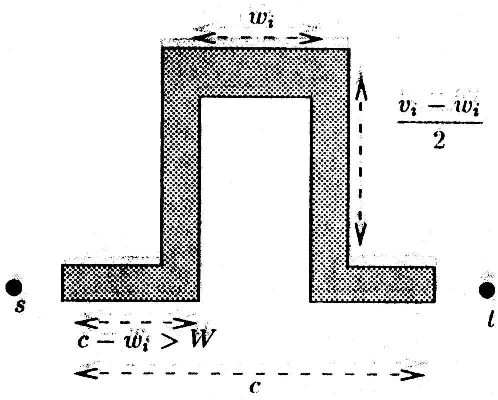

Figure 3: Blue tunnel thru red region for the ith item.