The Sum of Smaller Endpoint Degree over Edges of Graphs and Its Applications to Geometric Problems

(Extended Abstract)

Tatsuya Akutsu*, Yasukazu Aoki†, Susumu Hasegawa†,
Hiroshi Imai† and Takeshi Tokuyama‡

*Computer Science Division, Mechanical Engineering Laboratory
Namiki, Tsukuba, Ibaraki 305, Japan

†Department of Information Science, University of Tokyo
Hongo, Bunkyo-ku, Tokyo 113, Japan

‡IBM Research, Tokyo Research Laboratory
Sanban-cho, Chiyoda-ku, Tokyo 102, Japan

Abstract

In this paper we first show that, for a planar graph $G = (V, E)$ with vertex set V ($|V| > 22$) and edge set E,

$$\sum_{e=(u,v)\in E} \min\{\deg(u),\ \deg(v)\} \le 18|V| - 72,$$

where $\deg(v)$ is the degree of a vertex $v \in V$, and that this is tight. Also, for a planar graph having no cycle of length 3, the summation is shown to be at most $8|V| - 32$. This degree property can be used in the analysis of a computationally robust algorithm for Voronoi diagrams [7] and also to obtain another optimal randomized algorithm for finding the intersections among line segments and curves.

1. Introduction

In computational geometry, many geometric structures are represented as graphs, especially planar graphs, and there arise new graph problems in analyzing geometric algorithms. In this paper, we are interested in obtaining bounds of the sum of smaller endpoint degrees over edges of graphs, and utilize them to devise efficient algorithms for geometric problems.

For an undirected graph $G = (V, E)$ with vertex set V and edge set E, define $D(G)$ by

$$D(G) = \sum_{e=(u,v)\in E} \min\{\deg(u),\ \deg(v)\},$$

where $\deg(v)$ is the degree of a vertex v. Chiba and Nishizeki [2] show that $D(G)$ is at most $2a(G)|E|$ where $a(G)$ is the arboricity of G (the minimum number of trees covering G). Since the arboricity of a planar graph is at most 3, this upper bound for planar graphs becomes $6|E| \le 18|V| - 36$. This paper investigates $D(G)$ for planar graph G in more detail, and proves the following tight bounds.

Theorem. For a planar undirected graph $G = (V, E)$, which is simple and connected, with more than 22 vertices, $D(G) \le \min\{6|E| - 12,\ 18|V| - 72\}$. For a planar graph $G' = (V', E')$ with more than 14 vertices and without any cycle of length 3, $D(G') \le \min\{4|E'| - 8,\ 8|V'| - 32\}$. Also, there exist graphs G and G' satisfying $D(G) = 18|V| - 72$ and $D(G') = 8|V'| - 32$. □

This theorem can be used in the analysis of a computationally robust divide-and-conquer algorithm for constructing the Delaunay triangulation (Oishi and Sugihara [7]). Furthermore, this can be directly used to develop an optimal randomized algorithm for constructing the intersections (or arrangements) of line segments and curves. In this extended abstract, we provide only an outline of this algorithm. Although such optimal (randomized) algorithms are already known to exist (Chazelle, Edelsbrunner [1], Mulmuley [6]), this indicates the usefulness of the theorem in developing new geometric algorithms.

2. Upper Bounds

Let $G = (V, E)$ be a simple planar graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. A collection of pairs (v_i, E_i) $(i = 1, \ldots, n)$ of vertices v_i and edge subsets $E_i \subseteq E$ is called a $(1,c)$-matching if v_i is an endpoint of each edge in E_i, $|E_i| \leq c$ for each $i = 1, \ldots, n$, and

$$\bigcup_{i=1}^{n} E_i = E, \qquad E_i \cap E_j = \emptyset \ (i \neq j).$$

For $X \subseteq E$, define $V(X)$ to be the set of endpoints of edges in X. As is well known in network flow theory or transversal theory (e.g., see references in [5]), the existence of a $(1,c)$-matching is equivalent to the following condition:

$$|X| \leq c|V(X)| \qquad (\forall X \subseteq E).$$

For a graph with arboricity c, $|X| \leq c|V(X)| - c$ $(\emptyset \neq X \subseteq E)$ holds, which is used to derive a bound $2c|E|$ in [2].

Since G is a simple planar graph, Euler's relation states that, for any $X \subseteq E$ with $|X| \geq 2$,

$$|X| \leq 3|V(X)| - 6.$$

Hence, there exists a (1,3)-matching in G. Furthermore, in this case, it is seen that, for any pair of distinct edges (v_j, v_k), $(v_l, v_h) \in E$, there exists a (1,3)-matching (v_i, E_i) $(i = 1, \ldots, n)$ satisfying $E_j = \{(v_j, v_k)\}$, $E_k = \emptyset$, $\deg(v_j) \leq \deg(v_k)$, and one of $|E_l|$ and $|E_h|$ is 2 (e.g., see [5]). We then have

$$\begin{aligned} D(G) &= \sum_{i=1}^{n} \left(\sum_{e'=(u',v_i)\in E_i} \min\{\deg(u'),\ \deg(v_i)\} \right) \\ &\leq \left(3\sum_{i=1}^{n} \deg(v_i) \right) - 2\deg(v_j) - 3\deg(v_k) - \min\{\deg(v_l), \deg(v_h)\} \\ &= 6|E| - 2\deg(v_j) - 3\deg(v_k) - \min\{\deg(v_l), \deg(v_h)\}. \end{aligned}$$

Since G is connected, we can choose edges satisfying $\deg(v_j), \deg(v_k), \deg(v_l), \deg(v_h) \geq 2$, $D(G) \leq 6|E| - 12$.

We now show that $D(G) \leq 18|V| - 72$ when G is a maximal planar graph with more than 22 vertices. As is well known, any maximal planar graph with more than 12 vertices has a vertex with degree at least 6. Similarly, we can show that there exist at least 2 edges (u, v) with $\deg(u) + \deg(v) \geq 12$ for sufficiently large maximal planar graphs, say maximal planar graph with more than 22 vertices. Choosing these two edges as (v_j, v_k) and (v_l, v_h) and performing the case analysis, we can show that $D(G) \leq 6(3|V| - 6) - 36 = 18|V| - 72$. The detail will be given in the full paper. We thus have the upper bound for $D(G)$ in Theorem.

Note that this proof also indicates that

$$D(G) \leq 18|V| - 9 - 3\max_{v_i \in V} \deg(v_i).$$

Hence, if there is a vertex with large degree, $D(G)$ is much less than $18|V| - 72$ accordingly.

Now, suppose that G does not have any cycle of length 3. Then, Euler's relation states that, for any $X \subseteq E$ with $|X| \geq 2$,

$$|X| \leq 2|V(X)| - 4.$$

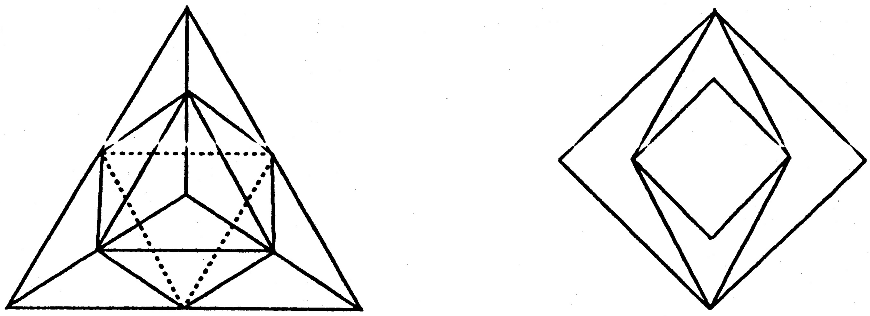
Applying the above arguments to this case, we can obtain the bound for G'. The detail will be described in the full version of this paper. Also, for this case, the proof also implies that

$$D(G') \leq 4|E| - 4 - 2\max_{v_i \in V} \deg(v_i).$$

3. Lower Bounds

We next consider lower bounds of the summation in the theorem for planar graphs.

Consider a regular tetrahedron T_0 whose edges are of unit length. Each face is a regular triangle, and there are 4 faces, 6 edges and 4 vertices of degree 3. By connecting the midpoints



(a) (b)

Figure 3.1. (a) T_1 and (b) S_2

of the edges, each triangle may be partitioned into four regular subtriangles; by repeating this process k times (denote the resultant polyhedron by T_i), each original triangle is divided into 4^k regular subtriangle with edges of length 2^{-k}. Figure 3.1(a) depicts T_1. Then, in the interior of each original face, there are $1 + 2^{2k-1} - 3 \cdot 2^{k-1}$ vertices of degree 6. Hence, in total, among $n = 4(1 + 2^{2k-1} - 3 \cdot 2^{k-1}) + 6(2^k - 1) + 4 = 2 + 2^{2k+1}$ vertices, only 4 vertices have degree 3 and the others 6, and no edge connects vertices of degree 3. Therefore, for such $n = 2 + 2 \cdot 4^k$ with $k = 2, 3, \ldots$, there exists a planar graph with n vertices for which the summation in the theorem is $18n - 72$.

To obtain a lower bound for planar graphs without cycle of length 3, construct the following series of graphs. S_0 is a square. We regard a pair of diagonal vertices in S_0 as new vertices. S_{i+1} is constructed from S_i by adding two new vertices and connect each of them with the new vertices in S_i. Figure 3.1(b) illustrates S_2. S_i has $n = 4 + 2i$ vertices, and $D(S_i) = 8n - 32$ for $i \geq 1$.

4. An Optimal Randomized Algorithm for Arrangements of Curves

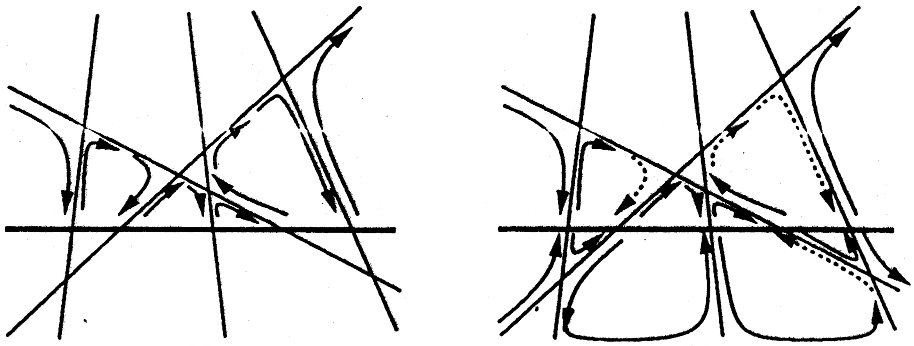
In this extended abstract, we will describe an $O(N^2)$-time randomized algorithm for constructing an arrangement of N lines, without using the zone theorem for lines. This illustrates, for the problem of constructing an arrangement of N curves such that any two curves intersect at a constant number of points, how a simple incremental algorithm using a careful search technique with $O(N^2)$ randomized time complexity may be devised based on the inequality in Theorem.

An incremental algorithm for constructing the arrangement of N lines $l_1, l_2, \ldots, l_N$ works roughly as follows: at the first stage, construct a trivial arrangement of one line l_1; at the ith stage ($i = 2, 3, \ldots, N$), add line l_i to the arrangement of lines $l_1, \ldots, l_{i-1}$, which has been computed already, to obtain the arrangement of lines $l_1, \ldots, l_{i-1}, l_i$. Here, the arrangement is represented by a standard data structure for planar subdivisions.

The main step here is to add l_i to the arrangement A_{i-1} of $l_1, \ldots, l_{i-1}$. To do this, we find an edge e of the arrangement A_{i-1} that is just above l_i at $x = -\infty$, and a cell c intersecting l_i at $x = -\infty$. This can be done in linear time by finding, from among the lines $l_1, \ldots, l_{i-1}$, the line of largest slope less than that of l_i. We then traverse A_{i-1} along l_i by following edges of the cell c in clockwise order, starting with e, to find a new intersection point of l_i with an edge e' of the cell. We iterate for $e := e'$ and $c :=$cell adjacent to c at e' until a cell is found intersecting l_i at $x = +\infty$. See Figure 4.1(a).

The time complexity of adding l_i to A_{i-1} is proportional to the number of edges of cells in A_{i-1} intersecting l_i (these cells form a zone of l_i, and this number is the complexity of the zone). The well-known zone theorem for lines (e.g., [3]) states that the complexity of this zone is $O(i)$. Hence, it takes $O(i)$ time to insert l_i to A_{i-1} to construct A_i, and in total the arrangement of N lines can be constructed in $O(N^2)$ time. Note that this time complexity is worst-case optimal, since the size of a simple arrangement is $\Theta(N^2)$.

In the above algorithm, of the cells intersecting l_i, only the portion above l_i is traversed. Instead of this, we may traverse edges of the upper and lower parts of a cell intersecting l_i one by one simultaneously so that a new intersection point of l_i with the cell may be found in time proportional to the length of the shorter of the two paths (upper and lower) from the old intersection



(a) (b)

Figure 4.1. (a) A simple incremental algorithm and (b) an incremental algorithm which chooses shorter paths

point to the new one. See Figure 4.1(b). This way of traversing cells is sometimes used in other geometric algorithms.

Now, let $e(l_j)$ be the number of edges traversed in adding l_j to the arrangement of lines $\{l_1,\ldots,l_i\} - \{l_j\}$ with choosing shorter paths as above ($j = 1,\ldots,i$). Consider the dual graph of the arrangement of $l_1,\ldots,l_i$ as a planar graph. This dual graph has at most i^2 edges, and does not have any cycle of length 3. Hence, applying the latter part of Theorem, it is seen that

$$\sum_{j=1}^{i} e(l_j) \leq 2 \cdot 4i^2.$$

By randomizing the order of insertion of lines in this modified incremental algorithm, the number of edges traversed in adding the ith line is at most $8i$ on the average. This implies that in total this algorithm constructs the arrangement of N lines in $O(N^2)$ average time.

This idea can be carried over for the case of the arrangement of curves, for which we need some of the techniques developed in [4,6], say the vertical decomposition of the arrangement. Besides these applications and that of [7], Theorem could be useful in the analysis of other geometric and graph problems.

Acknowledgment

The authors would like to thank Michael Houle for reading this paper and giving them comments. The work by Hiroshi Imai was partially supported by the Grant-in-Aid of the Ministry of Education, Science and Culture of Japan.

References

[1] B. Chazelle and H. Edelsbrunner: An Optimal Algorithm for Intersecting Line Segments in the Plane. *Proceedings of the 29th Annual IEEE Symposium on Foundations of Computer Science*, 1988, pp.590–600.

[2] N. Chiba and T. Nishizeki: Arboricity and Subgraph Listing Algorithms. *SIAM Journal on Computing*, Vol.14, No.1 (1985), pp.210–223.

[3] H. Edelsbrunner: *Algorithms in Combinatorial Geometry*. Springer-Verlag, 1987.

[4] H. Edelsbrunner, L. Guibas, J. Pach, R. Pollack, R. Seidel and M. Sharir: Arrangements of Curves in the Plane — Topology, Combinatorics, and Algorithms. *Proceedings of the 15th International Colloquium on Automata, Languages and Programming*, Lecture Notes in Computer Science, Vol.317, 1988, pp.214–229.

[5] H. Imai: Network-Flow Algorithms for Lower-Truncated Transversal Polymatroids. *Journal of the Operations Research Society of Japan*, Vol.26, No.3 (1983), pp.186–210.

[6] K. Mulmuley: A Fast Planar Partition Algorithm, II. *Proceedings of the 5th Annual ACM Symposium on Computational Geometry*, 1989, pp.33–43.

[7] Y. Oishi and K. Sugihara: Numerically Robust Divide-and-Conquer Algorithm for Constructing Voronoi Diagrams (in Japanese). *Transactions of the Information Processing Society of Japan*, to appear.