Watchman Routes for Multiple Guards

(Extended Abstract)

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Abstract

Given a simple polygon P, the Watchman Route Problem is to find the shortest closed path in P such that every point of P is visible from some point on the path. We consider the generalization of this problem to the case where more than one guard is available to patrol the polygon. We provide an efficient algorithm for the problem when P is a monotone rectilinear polygon and show that the general version of the problem is NP-Hard.

1 Introduction

In 1973, Victor Klee posed the problem of determining the minimum number of stationary guards needed to visually monitor the interior of any art gallery with n walls. Chvatal [2] and then Fisk [6] proved that $\lfloor n/3 \rfloor$ guards are sometimes necessary and always sufficient when the gallery is a simple polygon. A good survey of the work done on this problem and its many variations appears in O'Rourke's book, *Art Gallery Theorems and Algorithms* [11].

Variations of the Art Gallery Problem consider mobile guards which patrol along an edge, diagonal, or arbitrary line segment of the polygon. In these problems, one seeks the minimum number of mobile guards of the desired type that can cover the polygon, given that every point in the interior must be visible from at least one point on some guard's patrol route.

If the constraint that a mobile guard must travel along a single line segment is removed, then determining the minimum number of guards is trivial (one always suffices); in this setting, the natural objective is to minimize the length of a closed path from which a single guard can see the entire polygon. The resulting Watchman Route Problem is to provide an algorithm that computes such a path for each simple polygon given to it as an input.

Chin and Ntafos gave a linear-time algorithm for the Watchman Route Problem in a rectilinear polygon [4] and showed that the problem is NP-Hard for polygons with holes. They also provided an $O(n^4)$ time algorithm for the general case of a simple polygon when an "entry point", i.e., a point on the boundary that the route must pass through, is given [5]. The problem is still open for a simple polygon without an entry point.

Recently, Nilsson and Wood solved a multi-agent version of the watchman route problem in spiral polygons [10]. They gave an $O(mn^2)$ algorithm to find a collection of routes for m watchmen patrolling an n-sided spiral polygon such that the total length of all routes is minimized.

We feel that minimizing the length of the longest path traveled by any watchman is a more natural objective; it conveys the idea that we want to keep all routes short. With this ***mini-max*** objective, the longest time that any part of the gallery goes without surveillance is minimized. The ***min-sum*** objective used by Nilsson and Wood models the situation where security of the gallery is less important than the total energy expended by the watchmen and may be appropriate for the case of robot guards with high operating costs.

In this paper we consider the general mini-max multi-agent watchman route problem and show that it is NP-Hard. We also show that some restricted versions of it are NP-Hard. Finally, we provide a polynomial-time algorithm for this problem in a monotone rectilinear polygon using "rectangular" visibility.

2 Some NP-Hard Problems

In this section we show that the mini-max multi-agent watchman route problem and some restricted versions of it are NP-Hard. The general problem is stated below:

The Mini-Max Multi-Agent Watchman Route Problem

Instance: A simple polygon P and the number m of watchmen to be used.

Question: Is there a collection $\{p_i\}$ of m closed paths such that every point of P is visible to a point on some path p_i and the length of the longest path is $\leq K$?

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The reduction is from the following NP-Hard [9,11] problem:

Minimum Star Cover of a Simple Polygon

Instance: A simple polygon P, and a positive integer m.

Question: Is there a decomposition of P into m star-shaped subpolygons that cover (overlapping is permitted) it.

Remark: Minimum Star Cover (MSC) differs from the Art Gallery Problem (AGP) in that MSC asks for the minimum number of stationary guards that can cover a specific polygon while AGP asks for the minimum number of stationary guards sufficient to cover *any* n-sided polygon.

Theorem 1 *The Mini-Max Multi-Agent Watchman Route Problem is NP-Hard.*

Proof: Given an instance of Minimum Star Cover for which a cover of size m is sought, we create an instance of the Multi-Agent Watchman Route Problem with m watchmen and the maximum allowed path length equal to zero. Clearly, there exists a set of m watchman routes with maximum length equal to zero if and only if m *stationary* guards can cover the polygon. A covering of the polygon by star-shaped components could be obtained in polynomial time by constructing the visibility polygon of each point-sized watchman route. ∎

Unless **P** = **NP**, we have no hope of obtaining a polynomial-time algorithm for this problem. It is reasonable to conjecture that the complexity of the problem is polynomial in the size of the polygon but exponential in the number of watchmen. If this were true, we could solve the problem in polynomial time if we limit the number of watchmen *a priori* to some constant k. Unfortunately, this isn't true. We show below that even the 2-agent version of the problem is NP-Hard. The reduction is from the following NP-Complete problem [7]:

Partition

Instance: A set S of n objects with positive integer values $v(s)$.

Question: Is there a partition $S = S_1 \cup S_2$ such that $\sum_{s\in S_1} v(s) = \sum_{s\in S_2} v(s)$.

Theorem 2
The 2-Agent Mini-Max Watchman Route Problem is NP-Hard.

Proof: Given an instance of the partition problem, we generate in linear time a simple polygon with very narrow corridors whose lengths correspond to the values $v(s)$. Each corridor has a short recess protruding from its end at an angle, so a watchman must traverse the entire corridor to see into the recess. The longer path in any pair of watchman routes for this polygon has to have length at least $v = 1/2\sum_{s\in S} v(s)$. A longer path whose length equals v exists if and only if there exists a partition of S satisfying

$$\sum_{s\in S_1} v(s) = \sum_{s\in S_2} v(s) = v.$$

The desired partition could be obtained from the watchman routes in a straight-forward manner in linear time. ∎

Because the single-agent watchman route problem is easier in rectilinear polygons, one might suspect that the multi-agent watchman route problem is also easier in a rectilinear polygon. But the reduction given above can be modified to show that even the 2-agent problem is still NP-Hard in a rectilinear polygon.

Corollary 3 *The 2-agent mini-max watchman route problem in a rectilinear polygon is NP-Hard.*

3 Rectilinear Monotone Polygons

We showed in the previous section that the general Multi-Agent Mini-Max Watchman Route Problem is intractable. We also showed that the problem remains intractable when the number of watchmen is limited to two, even when the polygon is constrained to be rectilinear.

On the other hand, we mentioned previously that the Multi-Agent Watchman Route Problem can be solved efficiently in a spiral polygon. In an effort to find other special cases of the problem that admit efficient solutions, we turn now to the case of a rectilinear monotone polygon.

3.1 Preliminaries

A polygon P is a *rectilinear monotone polygon* (RMP) if its internal angles are all 90 or 270 degrees and it can be aligned so that all its edges are parallel to the x-axis or the y-axis and for every horizontal line h that intersects P, the intersection $h \cap P$ is a connected set. We assume without loss of generality that any such P has the orientation just described, i.e., it is monotone with respect to the direction of the y-axis. We can represent P by a set of two rectilinear chains each of which is monotone in the direction of the y-axis.

Now we summarize from [3] how to find the shortest watchman route for a single watchman in a rectilinear monotone polygon. We use this result to solve the subproblems that arise in the multi-agent version of the problem; in addition, we use some of their terminology in describing our algorithm.

A horizontal edge on the boundary of P is a *top edge* (*bottom edge*) if the interior of P lies below (above) it. Let $T(P)$ be the highest bottom edge of P and let $B(P)$ be the lowest top edge of P. Note that $P_T(P)$, the portion of P above $T(P)$, and $P_B(P)$, the portion of P below $B(P)$, are each star-shaped. Following Chin and Ntafos, we call the kernel of $P_T(P)$ the *top kernel* of P, and we call the kernel of $P_B(P)$ the *bottom kernel* of P.

Chin and Ntafos show that the shortest watchman route in a RMP traverses back and forth along a shortest path between the top kernel and the bottom kernel. They can find such a path in $O(n)$ time in a triangulated polygon. The recent linear-time triangulation algorithm of Chazelle [1] makes the overall complexity linear.

To make the multi-agent version of this problem tractable, we place a stronger requirement on the watchman than that each point of P has to be seen by at least one watchman. We require that each point must be seen by at least one watchman according to the following definition of visibility: two points p and q in P are *rectangularly visible* to one another if the unique rectangle with diagonal $\overline{pq}$ and with sides parallel to the x and y axes is fully contained in P. Rectangular visibility is a natural requirement in a rectilinear polygon and has appeared in the literature before; for instance, Keil [8] uses rectangular visibility in his algorithm for finding the minimum number of stationary guards needed to guard a rectilinear monotone polygon. Wood [12] gives a more complete discussion of rectangular and other alternative forms of visibility.

We define the *traversed region* of a watchman route r to be the subpolygon of P between the horizontal lines through the highest and lowest points of r. By monotonicity, every point in the traversed region of a watchman route is visible from a point on that watchman route with the same y-coordinate.

3.2 Properties of Optimal Watchman Routes

In this section we characterize optimal watchman routes so that we can find them efficiently. The important qualities of an optimal set of watchman routes are summarized in the following propositions; most of these propositions depend on the fact that we are using rectangular visibility.

Proposition 1 *In an optimal set of watchman routes for a RMP P, each edge of the boundary of P is seen in its entirety by at least one watchman.*

Proposition 2 *In an optimal set of watchman routes, the routes can be adjusted so that everything seen by a route above its traversed region can be seen from the point(s) of the route having the maximum y-coordinate. Similarly, everything seen below the traversed region of a route can be seen from the point(s) of the route having the minimum y-coordinate.*

Proposition 3 *In an optimal set of watchman routes the traversed regions of any two routes are disjoint.*

Proposition 4 *The watchman routes in an optimal set of m watchman routes can be adjusted so that for every pair of watchman routes r_i and r_{i+1} in a sequential ordering of the watchman routes from top to bottom, the region of P between the traversed region of r_i and the traversed region of r_{i+1} can be completely seen from routes r_i and r_{i+1}. Also, the region of P above r_1 can be seen from r_1 and the region below r_m can be seen from r_m.*

Proposition 5 *In between two sequential traversed regions, the left and right chains of P have the following structure from top to bottom: the left chain consists of a staircase chain monotonically increasing in x, a vertical transition edge, and a staircase chain monotonically decreasing in x; the right chain consists of a staircase chain monotonically decreasing in x, a vertical transition edge, and a staircase chain monotonically increasing in x.*

Proposition 6 *In between two sequential routes, no edge that is below the vertical transition edge of its chain can be seen from above and no edge above the transition edge can be seen from below.*

Proposition 7 *The region of P between two sequential watchman routes can be partitioned, into a region seen from above and a region seen from below, either by a line segment connecting two vertices on opposite chains of P which are (rectangularly) visible to one another or by a horizontal segment from a vertex of P to the opposite chain.*

3.3 The Algorithm

We make two important conclusions from the long string of propositions given in the previous section. First, an optimal set of m watchman routes in a RMP P can be found by finding an "optimal" partition of P into m disjoint pieces and then constructing the shortest single-agent watchman route in each piece. And second, each of the subpolygons arising from a valid set of $m-1$ cut edges is either a monotone rectilinear polygon or can be made into one by replacing each of its non-horizontal cut edges by two edges of the rectangle the cut edge is a diagonal of. This replacement has no effect on the length of the shortest watchman route that sees all the original edges of the subpolygon.

The discussion above suggests that if we solve a number of single-agent RMP watchman route problems, we can use the results to find an optimal set of watchman routes for several agents. This idea lets us formulate the problem recursively and solve it efficiently using Dynamic Programming.

The recursive formulation follows. Denote the highest horizontal edge of P by e_T. Order the valid cut edges e_i lexicographically according to the y-coordinates of their left and right endpoints. Define a partial order $\prec$ on the valid cut edges as follows: $e_j \prec e_i$ iff the y-coordinate of each endpoint of e_j is less than or equal to the corresponding y-coordinate of e_i. Let V denote the mini-max value of an optimal set of m watchman routes. For any pair of edges e_i and e_j with $e_j \prec e_i$, denote the length of the shortest single-agent watchman route in the RMP determined by e_i and e_j by $V_1(e_i, e_j)$. Also denote by $V(e_i, k)$ the minimax value of an optimal set of k watchman routes in the portion of P below the cut e_i. Then the optimal value V can be expressed as

$$V = \min_i \{\max\{V_1(e_T, e_i), V(e_i, m-1)\}\}.$$

The term $V(e_i, m-1)$ can also be expressed recursively as

$$V(e_i, k) = \min_{e_j \prec e_i} \{\max\{V_1(e_i, e_j), V(e_j, k-1)\}\}.$$

The first step of the algorithm precomputes $V_1(e_i, e_j)$ for each pair (e_i, e_j) with $e_j \prec e_i$. Since there are $O(E)$ valid cut edges, where E is the size of the (rectangular) visibility graph of P, we need to solve $O(E^2)$ subproblems and store the results in a table of size $O(E^2)$. The straight-forward approach to this initialization step takes $O(n)$ time to calculate each entry in the table. A more careful analysis, however, enables us to compute the entire table in time $O(En^2)$.

After initializing the table for $V_1(e_i, e_j)$, we need to compute a one-dimensional table of size $O(E)$ corresponding to $V(e_i, k)$ for each $k < m$. By computing the tables sequentially for k from 2 to $m-1$, we can compute each entry in $O(E)$ time by comparing $O(E)$ numbers which are obtained in constant time from the previously computed tables. Thus the total time needed to construct the tables and evaluate V is $O(E^2 m)$. Once we have calculated V, we know an optimal set of $m-1$ cut edges and can easily compute an optimal set of m watchman routes.

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