# Illuminating Squares on a Transversal

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Hoffman has shown that $\lfloor n/4 \rfloor$ lights are sufficient to illuminate the interior of an orthogonal polygon with $n$ vertices and $h$ orthogonal holes [5]. This implies that $n+1$ lights are sufficient to illuminate $n$ squares in the plane. Here we show that in the case that the set of squares admits a transversal $\lfloor 2n/3 \rfloor + 2$ lights are sometimes necessary and that $\lfloor 2n/3 \rfloor + 7$ are always sufficient. We also give an $O(n \log n)$ time algorithm for finding a placement of such lights. These results answer a question posed by Toussaint at the 1990 Bellairs Workshop on Illuminating Sets.

## 1 Preliminaries

A line which intersects each member of a set $S$ of objects is called a *line transversal* for $S$. A line is a *proper line transversal* if it intersects each object in $S$ in more than one point. From now on the term transversal means proper line transversal.

A *point-light-source*, or just *light* for short, is assumed to emit rays of light in all directions in straight lines. *Free space*, denoted $FS$, is the plane minus all of the objects in the plane. A point $p$ in $FS$ is *illuminated* if there is some light such that the line segment from $p$ to the light does not intersect the interior of any object. A set of objects $S$ is said to be illuminated if all points in $FS$ are illuminated.

A *manhattan skyline* $M = (p_1, \ldots, p_n)$ is an infinite rectilinear monotone polygonal chain. (Note that the points are ordered.) $M$ partitions the plane into two regions both of which we will call skies. We present the following lemma.

**Lemma 1.1** *Any sky in $FS$ can be illuminated by two lights. Furthermore, finding positions for these two lights can be done in $O(n)$ time.*

**Sketch of Proof:** To identify two feasible positions for illuminating the upper sky, first compute the following five values: $x_M$ = maximum $x$-coordinate; $x_m$ = minimum $x$-coordinate; $y_M$ = maximum $y$-coordinate; $y_m$ = minimum $y$-coordinate; and $h_m$ = minimum length of a horizontal edge. One light may be positioned at $(x_0, y_0)$ where $x_0 = x_m - 1$ and $y_0$ satisfies:

$$\frac{y_M - y_m}{\frac{1}{2}h_m} = \frac{y_0 - y_m}{x_M - x_0}.$$

The position for the second light is defined symmetrically. □

## 2 Main Theorem

Let $S = \{s_1, \ldots, s_n\}$ be a set of $n$ isothetic unit squares in the plane which admit a

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transversal. Assume the transversal is between 0 and 45 degrees. A point is said to be a *blocked point* if it cannot be translated to infinity in any of the four axial directions without hitting a square. Space limitations prevent a more formal proof of the following lemma.

**Lemma 2.1** $\lfloor 2n/3 \rfloor + 2$ *lights are necessary to illuminate a set $S$ of $n$ squares on a transversal.*

**Sketch of Proof:** Consider the example in Figure 1. To illuminate all the blocked points, indicated by shaded regions in the figure, $\lfloor 2(n-9)/3) \rfloor + 2 = \lfloor 2n/3 \rfloor - 4$ lights are required. Each of these lights can illuminate at most one of the vertical strips. There are $\lfloor 2n/3 \rfloor - 2$ of these so at least two of these are not illuminated. Also, there are four horizontal strips which are not illuminated. At least six lights are required to illuminate these two vertical strips, these four horizontal strips and the exteriors of the remaining squares. □

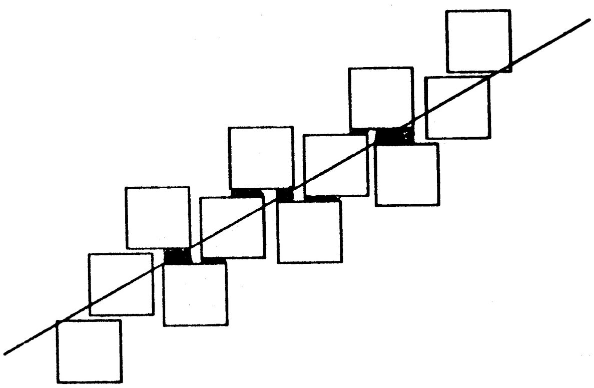

Figure 1: Shaded areas indicate blocked points.

For the proof of sufficiency the idea is to show that $\lfloor 2n/3 \rfloor - 1$ lights are sufficient to illuminate all blocked points and that 8 lights are sufficient to illuminate the rest of the plane.

We say that a square $s$ is ***above*** a blocked point if the point hits $s$ when translated directly upward. Similarly $s$ may be ***below***, to the ***left*** or to the ***right*** of a blocked point. Every blocked point has at least one square above, below, to the left and to the right of it. We are going to illuminate each blocked point by putting a light on the lower side of one of the squares above it or on the upper side of one of the squares below it.

**Lemma 2.2** *A light placed on the lower (upper) side of a square $s$ is sufficient to illuminate all blocked points which are below (above) $s$.*

**Proof:** We prove the first statement; the second proof is similar. Let $p$ be a blocked point below $s$ at maximum vertical distance from $s$. Let $q$ be the projection of $p$ onto the bottom side of $s$. Place a light $l$ at $q$. Suppose there is some blocked point $r$ below $s$ that is not illuminated by $l$. Then there must be some square which intersects line segment $rq$. Any square which intersects $rq$ is either above $r$ contradicting the fact that $s$ is above $r$ or it also intersects $pq$ contradicting the fact that $s$ is above $p$. □

Call a square a TOP square if the transversal goes through its left and top sides; call it a MID square if the transversal goes through its left and right sides; call it a BOTTOM square if the transversal goes through its bottom and right sides. Since we insist on a proper transversal, the transversal cannot go through the top left corner or the bottom right corner of any square. If the transversal goes exactly through the bottom left corner then call it a MID square. Since we assume that the transversal is between 0 and 45 degrees these are the only possibilities.

**Lemma 2.3** *If a square is above a blocked point then it is either a BOTTOM square or* *a MID square. If a square is below a blocked point then it is either a MID square or a TOP square.*

**Proof:** We show the result for squares above blocked points; the argument for squares below blocked points is similar. Suppose there is a TOP square $s$ above a blocked point. Consider the right square $r$ for this blocked point. $r$ must lie entirely to the right of the line passing through the left side of $s$ otherwise there can be no blocked point below $s$. Thus, since $r$ must intersect the transversal and since the transversal intersects the top side of $s$, $r$ must lie entirely above the bottom side of $s$. But then it cannot be the right square of the blocked point. □

**Lemma 2.4** *A blocked point cannot have a MID square above it and a MID square below it.*

**Proof:** Let $a$ be a MID square above a blocked point. Suppose $b$ is a MID square below this blocked point. Since $a$ and $b$ are MID squares, they must be vertically separable. Hence $b$ cannot be below the blocked point. □

**Lemma 2.5** *All the blocked points can be illuminated with* $\lfloor 2n/3 \rfloor - 1$ *lights.*

**Proof:** By Lemmas 2.3 and 2.4 the possibilities for a blocked point is that it has (1) a BOTTOM square above it and a MID square below it, or (2) a MID square above it and a TOP square below it, or (3) a BOTTOM square above it and a TOP square below it. All the blocked points get illuminated if we place lights either (a) above all TOP squares and above all MID squares, or (b) below all MID squares and below all BOTTOM squares, or (c) above all TOP squares and below all BOTTOM squares. Each square is either a TOP, MID or BOTTOM square. One of the sets of TOP, MID, or BOTTOM squares contains $\lceil n/3 \rceil$ or more of the squares. If there are $\lceil n/3 \rceil$ or more BOTTOM squares then all blocked points can be illuminated with $\lfloor 2n/3 \rfloor$ lights using strategy (a). Similarly for the other two cases. That each blocked point is illuminated follows from Lemma 2.2. In fact, we can reduce the number of lights required by one by observing that the leftmost and the rightmost squares hit by the transversal cannot be above or below any blocked point. Thus we have $\lfloor 2(n-2)/3 \rfloor \leq \lfloor 2n/3 \rfloor - 1$ lights are sufficient to illuminate all the blocked points. □

**Lemma 2.6** 8 *lights are sufficient to illuminate the plane minus the blocked points.*

**Proof:** Free space minus the blocked points can be partitioned into 4 manhattan skylines each of which can be illuminated by 2 lights by Lemma 1.1. □

Now the theorem follows from Lemmas 2.1, 2.5 and 2.6.

**Theorem 2.7** $\lfloor 2n/3 \rfloor + 1$ *lights are sometimes necessary and* $\lfloor 2n/3 \rfloor + 7$ *lights are always sufficient to illuminate a set* $S$ *of* $n$ *squares on a transversal.*

## 3 The Algorithm

Positioning the $\lfloor 2n/3 \rfloor + 7$ lights can be done in $O(n \log n)$ time as follows. First sort the squares by $x-$*coordinate* in $O(n \log n)$ time and then in linear time find a line transversal [2]. In $O(n)$ time we can decide which lighting strategy to use for the blocked points. Using the next-element-subdivision method of Edelsbrunner, Overmars and Seidel [3] we can compute the set of blocked points above and below each square in $O(n \log n)$ time. Placing all of the $\lfloor 2n/3 \rfloor - 1$ lights can be done in time proportional to the size of the next-element subdivision which is $O(n)$ [3]. The next-element-subdivision also provides us with the four manhattan skylines. Now the remaining 8 lights can then be positioned in $O(n)$ time by Lemma 1.1.

# 4 Open Problems

How many lights are necessary and sufficient to illuminate a set of $n$ regular $k$-gons which admit a transversal? Here we have shown the result for $k = 4$. Czyzowicz, Rivera-Campo and Urrutia have shown that $n + 1$ are sufficient to illuminate $n$ homothetic triangles [1]. Fejes-Toth has shown that $2n - 2$ lights are sufficient to illuminate $n$ convex disks which admit a transversal [4].

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