Finding All the Largest Circles in a 3-dimensional Box

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Extended Abstract

The problem of finding the largest 1-dimensional hypersphere, i.e., line segment, in a d-dimensional convex polyhedron is just the problem of finding the diameter of the polyhedron. This problem is easily solved in $O(n^2)$ time, where n is the number of vertices in the polyhedron. The problem of finding the largest d-dimensional hypersphere in a d-dimensional convex polyhedron can be formulated as a linear program and solved in polynomial time [2]; in particular, it can be solved in linear time when d is fixed [3]. For $d = 2$ a different linear time algorithm is presented in [1]. Victor Klee has posed the problem of finding the largest $(d-1)$-dimensional hypersphere in a d-dimensional convex polyhedron. In particular the problem of finding the largest circle in a 3-dimensional convex polyhedron is open. In this abstract show how to find all the largest circles in a 3-dimensional box. Theorems 2.1 and 2.2 give the main results.

1 Preliminaries

Given three pairs of parallel planes, each pair orthogonal to the other two, a *box* B is the intersection of the closed regions between the pairs of planes. Let $front$, $back$, $left$, $right$, *top* and *bottom* denote the six planes determining B. We call these the *face planes* of B. Without loss of generality, assume $front$ and *back* are parallel to the yz-plane, $left$ and $right$ are parallel to the xz-plane and *top* and *bottom* are parallel to the xy-plane. We will sometimes refer to the xy-plane, the yz-plane and the xz-plane as the *axis planes*. We assume that B is centered at the origin o and is specified by the coordinates (a, b, c) of the intersection of $front$, *top* and $right$. Hence B has dimensions $2a \times 2b \times 2c$. Without loss of generality assume $0 < a \leq b \leq c$. See Figure 1.

Let C^* denote a circle (not necessarily unique) of largest radius contained within B. Thus our problem is to find the radius r^* of C^*. Let π^* denote the plane containing C^*. It can be shown that C^* can be chosen such that plane π^* is not parallel to any axis plane. Plane π^* is determined by its unit outward normal vector $m^* = (x^*, y^*, z^*)$ and by its distance d^* from the origin. In other words $\pi^* = \{(u, v, w) | (u, v, w) \cdot (x^*, y^*, z^*) = d^*\}$.

Assumption: *From now on, we assume that C^* has been chosen such that π^* is not parallel to an axis plane.*

Notation: *We let π denote any plane not parallel to an axis plane such that $\pi \cap B \neq \emptyset$.*

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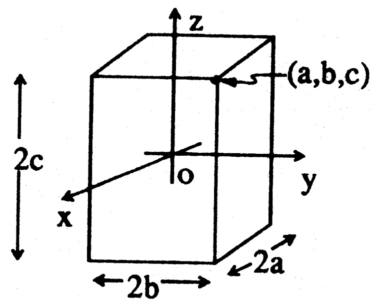

Figure 1: Box B.

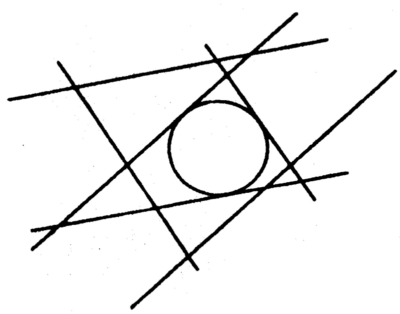

Figure 2: The radius of C may be determined by three lines.

Observation 1.1 *The intersection of a plane π with a pair of parallel planes is a pair of parallel lines whose distance apart is invariant under translations of π.*

Since the face planes of B comprise three pairs of parallel planes the intersections of a plane π with the face planes of B determine three pairs of parallel lines on π. Let the distances between these pairs of parallel lines be denoted by $d_{front,back}(\pi)$, $d_{left,right}(\pi)$ and $d_{top,bottom}(\pi)$. By Observation 1.1, that these distances do not depend on the distance of π from the origin.

Observation 1.2 *A circle in a plane π not parallel to an axis plane lies entirely within B if and only if it lies between the pairs of parallel lines on π determined by the intersection of π with the parallel face planes of B.*

Hence the distance between the closest pair of parallel lines formed by the intersection of π with the face planes of B gives an upper bound on r, the radius of a largest circle C in $B \cap \pi$. Note that it is conceivable that C does not touch both lines of any parallel pair (see Figure 2), so the radius of C is not necessarily equal to the distance between the closest pair of parallel lines. Nevertheless, we have the following key observation.

Key Observation *Consider a family of planes intersecting B but parallel to no axis plane. Then the supremum, over all planes* π *in the family of*

$$\frac{1}{2}\min\{d_{front,back}(\pi), d_{left,right}(\pi), d_{top,bottom}(\pi)\}$$

is an upper bound for the radius of the largest circle contained in B and lying in a plane in the family.

For some families of planes, the supremum mentioned in the Key Observation could be achieved by a plane in the family and hence could be a maximum. The intersection of a plane passing through the origin with a pair of parallel face planes of B is a pair of parallel lines equidistant from the origin. Thus, by Observation 3.1:

Observation 1.3 *Suppose a plane* π *passes through the origin. Then* $\pi \cap B$ *contains a circle of radius* r *centered at the origin if and only if* r *is at most half the distance between the closest pair of parallel lines determined by the intersection of* π *with a pair of parallel face planes.*

Intuitively, the following observation is equivalent to saying that it is impossible to adjust a unit vector so as to simultaneously decrease the lengths of its projections onto all the axis planes:

Observation 1.4 *Let* $m = (x, y, z)$ *be a unit vector. Then there is no other unit vector* $m' = (x', y', z')$ *such that* $\sqrt{x^2+y^2} > \sqrt{x'^2+y'^2}$, $\sqrt{y^2+z^2} > \sqrt{y'^2+z'^2}$ *and* $\sqrt{x^2+z^2} > \sqrt{x'^2+z'^2}$.

2 Finding a largest circle

We want to find the plane π^* that contains C^*, where by our Assumption, π^* is not parallel to an axis plane. Our approach is to determine upper bounds for the radius of the largest circle enclosed in B that lies in each of the following two families of planes:

(1) planes intersecting B and parallel to exactly one axis, and

(2) planes intersecting B and parallel to no axis.

Then we show that the larger of these upper bounds, which turns out always to be the upper bound obtained for the family in (2), can be realized as the radius of a circle enclosed in B. (The value of the bound obtained for (2) depends on whether $c^2 \geq a^2 + b^2$.) To find the upper bounds for (1) and (2), we use the Key Observation, computing

$$\sup\{\min\{d_{front,back}(\pi), d_{left,right}(\pi), d_{top,bottom}(\pi)\}\},$$

where π ranges over the planes in categories (1) and (2), respectively.

Theorem 2.1 *The radius* r^* *of the largest circle contained in B is given by*

$$r^* = \begin{cases} \sqrt{a^2+b^2}, & \text{if } c^2 \geq a^2+b^2; \\ \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{2}}, & \text{if } c^2 < a^2+b^2. \end{cases}$$

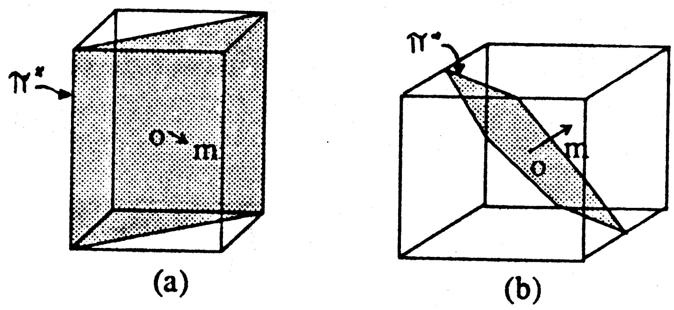

Figure 3: (a) π^* when $c^2 \geq a^2 + b^2$ (b) π^* when $c^2 < a^2 + b^2$.

Proof Sketch: In case $c^2 \geq a^2 + b^2$, the plane through the origin and a diagonal of the *top* of B intersects B in a $2c \times 2\sqrt{a^2+b^2}$ rectangle whose vertices are $(-a, b, c)$, $(a, -b, c)$, $(-a, b, -c)$, $(a, -b, -c)$. This rectangle contains a circle of radius $\sqrt{a^2+b^2}$, which is equal to the larger of the upper bounds obtained for families (1) and (2). See Figure 3(a).
In case $c^2 < a^2 + b^2$, we consider the plane π through the origin having unit outward normal

$$m = \left(\pm\sqrt{\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}}, \pm\sqrt{\frac{a^2-b^2+c^2}{a^2+b^2+c^2}}, \pm\sqrt{\frac{a^2+b^2-c^2}{a^2+b^2+c^2}} \right).$$

For this plane,

$$\frac{d_{front,back}(\pi)}{2} = \frac{d_{left,right}(\pi)}{2} = \frac{d_{top,bottom}(\pi)}{2} = \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{2}},$$

which equals the larger of the upper bounds obtained for families (1) and (2) in this case. By Observation 1.3, $\pi \cap B$ contains a circle of this radius centered at the origin. See Figure 3(b). □

Theorem 2.2 *The largest circle contained in B is unique up to the symmetries of B unless* $c^2 > a^2 + b^2$. *In this case, the set of largest circles consists of those circles centered at* $\{(0,0,z) \mid -c + \sqrt{a^2+b^2} \leq z \leq c - \sqrt{a^2+b^2}\}$ *lying on a plane through the origin and containing a diagonal of the top face.*

References

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