Computing Minkowski Sums of Regular Polygons

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1 Introduction

The Minkowski sum of two sets $M, N \subset \mathbf{R}^d$, denoted $M \oplus N$, is defined as the set $\{p+q : p \in M, q \in N\}$. If for any set S, $-S$ denotes $\{-q : q \in S\}$ and S_p denotes $\{p+q : q \in S\}$, then it is easy to see that $M \oplus N = \{p : M \cap (-N)_p \neq \emptyset\}$. §

In this paper, we consider Minkowski sums of polygons that are path-connected, bounded and regular in $\mathbf{R}^2$, because of their widespread use in solid modelling. For brevity, we will often refer to them simply as polygons. It is easy to see that the Minkowski sum of path connected, bounded and regular polygons in $\mathbf{R}^2$, remain path connected, bounded and regular, so that this class of objects is closed under Minkowski summation.

Any polygon can be decomposed into a disjoint collection of open, connected subsets of the plane, open line segments and points, such that the boundary of each of these open sets is the union of some of these segments and points and that the boundary of each of the segments is two of the points. We assume that each polygon has such a "clean decomposition" and refer to the open sets as 2-cells, the open line segments as 1-cells (or edges) and the points as 0-cells (or vertices).

Let S be a polygon with a clean decomposition and define the size of S, $|S|$, to be $s_0 + s_1 + s_2$, where s_0 is the number of vertices, s_1 is the number of edges and s_2 the number of 2-cells in the decomposition of S. Since each 1-cell bounds at most two 2-cells, $2s_1 \geq s_2$. Since each 0-cell bounds a 1-cell, $2s_1 \geq s_0$. This implies that $\mathbf{O}(|S|) = \mathbf{O}(s_1)$. In fact, it can be shown that $\mathbf{O}(s_1) = \mathbf{O}(s_0)$.

An important part of designing and analyzing algorithms is knowledge of the output size, as a function of the size of the input. Other authors [3, 5] have noted that the Minkowski sum of two convex polygons in $\mathbf{R}^2$, with m and n edges respectively, can be computed by merging their edges in slope order and that this leads directly to a tight asymptotic bound on the output size of $\mathbf{O}(m+n)$. On the other hand, if one polygon is convex, with m edges, and a second polygon is non-convex, with n edges, [2] observed that the output size of the Minkowski sum has a tight asymptotic bound of $\mathbf{O}(mn)$. Little appears in the literature for the case involving two non-convex polygons.

In this paper, we will determine the worst case asymptotic complexity of the output size of the Minkowski sum of general polygons and develop an algorithm for the computation of the sum. In particular, Section 3 discusses an algorithm for computing the Minkowski sum of regular polygons. Section 3.2.2 discusses the output size of Minkowski sums of non-convex polygons, and results of this section are summarized in Table 1.

2 Preliminaries

If E is a 1-cell in the boundary of a polygon P and n is a vector that is perpendicular to E such that for each $p \in E$, there exists $\delta_0 > 0$ such that $p \oplus \delta n$ is contained in the complement of P, for all $0 < \delta < \delta_0$, we will call n an *exterior normal* for E (with respect to P). Every such 1-cell has an exterior normal and up to a multiplication by a positive scalar it is unique. If p is a point in E, then the exterior normal at p is defined to be the exterior normal to E.

A polygon P is *convex* at point $p \in \partial P$ if there exists a line l, which contains p, such that P is contained in one of the halfplanes defined by l. A polygon P is *locally convex* at point $p \in \partial P$ if there exists an open set O

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§ Basic definitions and properties of Minkowski sums can be found in [11].

M Convex, N Convex	M Convex, N NonConvex	M NonConvex, N NonConvex
$O(m+n)$	$O(mn)$	$O(m^2n^2)$

Table 1: Worst case upper bounds for $\mid M \oplus N \mid$, where $m = \mid M \mid$ and $n = \mid N \mid$. These bounds are sharp.

that contains p such that $P \cap O$ is convex at p. If P is locally convex at a vertex v of P, then there exists two 1-cells, e_1 and e_2 in ∂P, for which v is a vertex such that the convex hull of e_1 and e_2 contains $P \cap O$, whenever O is sufficiently small. We define the *cone of normals* at v to be the smaller, closed angular region determined by the exterior normals to e_1 and e_2. If C is the cone of normals at v, then $C \cap (O \cap P) = p$.

If p is any point on the boundary of a polygon P, then define the *normal set* of p, denoted N_p, to be the exterior normal to p if p is a point in a 1-cell of ∂P, the cone of normals to p if p is a locally convex 0-cell of P, and $\emptyset$ otherwise.

The normal set of a point p of the ∂P is useful because of the following. If l is any line containing p which is perpendicular to a non-zero vector in the set N_p, then $P \cap O$ belongs to one of the half planes determined by l, for any sufficiently small open set O containing p. Conversely, if l is any line containing p which is perpendicular to no non-zero vector in the set, then each half plane determined by l contains a non empty open subset of $O \cap P$ for any open set O containing p.

For polygons P and Q, $p \in \partial P$ is a *supporting point* for $q \in \partial Q$ if $N_p \cap N_q \neq \{0\}$.

Proposition 1 *For polygons M and N, every point $p \in \partial(M \oplus N)$ is contained in the Minkowski sum of the closure of an edge of M and a supporting vertex of N or the Minkowski sum of the closure of an edge of N and a supporting vertex of M.*

The proof is omitted for brevity.

It can be seen that not all combinations of boundary-point pairs will result in points on the boundary of the resulting Minkowski sum.

3 Algorithm

In this section we shall present an algorithm to perform the Minkowski addition of two polygons. We shall be utilizing Proposition 1 to generate a superset set of the boundary of the final result. We shall compute the Minkowski sum of the edges of each polygon and the corresponding supporting points for that edge among the vertices of the other polygon. We are guaranteed that the set of edges produced will be a superset of the edges that comprise the boundary of the Minkowski sum of the two polygons. As each edge in the final result has been derived from the Minkowski sum of an edge with a vertex, it implies that the final edge is nothing but a translated version of the initial edge. It therefore retains its orientation and material direction.

From the superset of edges we can compute the arrangement and determine all 2-cells in the arrangement. These 2-cells can be classified on the basis of their material direction. All 2-cells that can be definitively classified as lying "In" the Minkowski sum will be marked as such. For those 2-cells that cannot be easily classified, we will select any point in the interior of the 2-cell and classify it using the definition of Minkowski sum as detailed in Section 1. The classification of the point can be propagated throughout the 2-cell to which the point belonged. Once all the 2-cells have been classified, we can merge similarly classified 2-cells across boundaries.

3.1 Statement of the Algorithm

The algorithm is stated below:

Input : Regular polygons M and N.

Output: M $\oplus$ N.

1. Get a superset E of the boundary edges
   (a) Set $\Gamma = \{\emptyset\}$
   (b) For each edge E in ∂M
   for each vertex V in ∂N that locally supports E, add $E \oplus V$ to Γ.
   (c) For each edge E in ∂N
   for each vertex V in ∂M that locally supports E, add $E \oplus V$ to Γ.
2. Compute the full arrangement A of the edges in Γ.
3. Set the classification of all 2-cells in A as "unknown".
4. For each 2-cell C in A
   If any boundary edge of C has a neighborhood in C, then classify C as "in".
5. For each 2-cell C in A that is still classified as "unknown"
   (a) Pick a point P_C in the interior of C.
   (b) If $N \cap (-M)_{P_C} = \emptyset$ then classify C as "out"
   else classify C as "in"
6. Cleanup: Merge adjacent 2-cells that have the same classification and have a common boundary edge.

Details of the Algorithm are omitted for brevity. They can be found in [7]

3.2 Analysis of the Algorithm

3.2.1 Complexity

To compute the worst case asymptotic complexity of the algorithm, let us look at its various parts. Let the two input polygons be M and N. In step 1 we can generate the superset of the boundary, by converting the problem to a range searching problem. This problem can be solved using interval trees [10] in $\mathbf{O}(m \ln m + n \ln n + s)$, where s is the number of edges generated in the superset. In the worst case s can be $\mathbf{O}(mn)$. Computing the full arrangement can be done in $\mathbf{O}(s + k)\ln(s)$, where k is at worst $\mathbf{O}(s^2)$. Also the number of 2-cells generated in the arrangement is at worst $\mathbf{O}(k)$. The final classification step can be done in $\mathbf{O}(k(m+n)\ln(m+n))$. As a result the complexity of the algorithm is $\mathbf{O}(m \ln m + n \ln n + s + (s+k)\ln s + k(m+n)\ln(m+n))$. Clearly in the worst case when s is $\mathbf{O}(mn)$ and k is $\mathbf{O}(m^2n^2)$, the worst case asymptotic complexity is $\mathbf{O}(m^2n^2(m+n)\ln(m+n))$.

If we study the complexity of the algorithm for the different classes of inputs, we find that for polygons obtained as a result of the Minkowski sum of two convex polygons, the complexity is $\mathbf{O}((m+n)\ln(m+n))$, which is within a log factor of the optimal. For Minkowski sums of a convex with a non-convex polygon, the complexity is $\mathbf{O}(mn(m+n)\ln(m+n))$, where the output size is $\mathbf{O}(mn)$.

3.2.2 Output Size

From the algorithm we see that the worst case asymptotic complexity is governed by the number of 2-cells generated as a result of the arrangement. We can easily see that for non-convex polygons M and N, of sizes $\mathbf{O}(m)$ and $\mathbf{O}(n)$ resp., the number of edges in the superset can be $\mathbf{O}(mn)$. Also if each edge in the superset intersects all the other edges, then the arrangement of the superset will have $\mathbf{O}(mn)^2$ edges. If each of these edges appears on the boundary then the output will also be $\mathbf{O}(mn)^2$. This bound is easily attainable for the class of polygons, an example of which is shown in Figure 1. For these polygons, the dimensions are so chosen such that, the final result of the Minkowski sum of the two polygons has $\mathbf{O}(mn)^2$ edges.

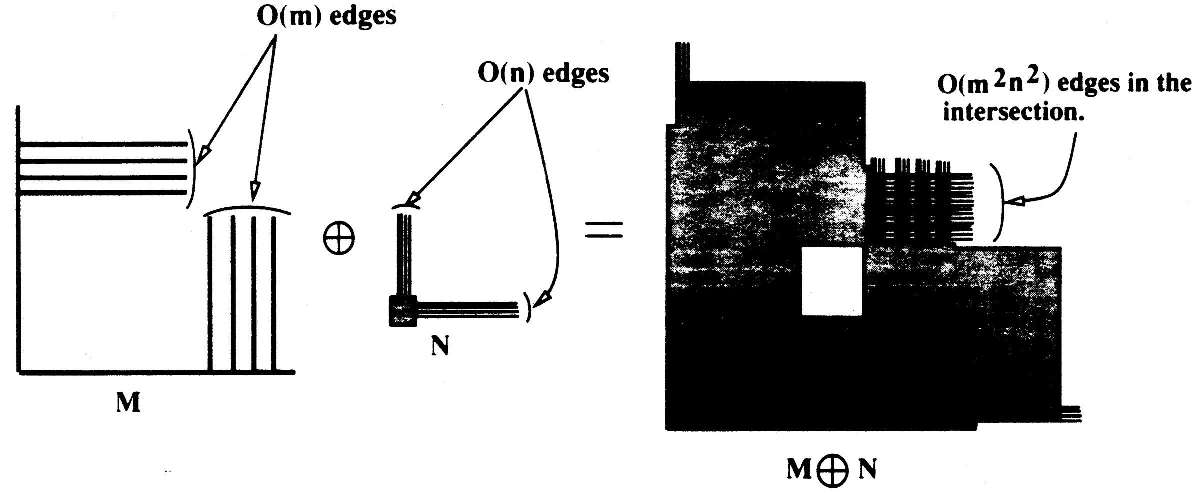

Figure 1: NonConvex polygon $M \oplus$ NonConvex polygon N

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