On A Class of 2-D Compliant Motion Planning Problems

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Abstract

In this paper, a class of 2-D compliant motion planning problems [1] [2], namely, a disc pushing another disc, is studied. The space is shown to possess some unusual and complex properties not seen in other motion planning problems. We characterize the 2-D space by dividing it into six different types of regions with different dimension, symmetry and directedness. The boundaries between different regions are then characterized and complexity of motion planning in each type of region is discussed.

Consider a system consisting of two circular discs A and B in a closed 2-D space (which we call a *room*) with a polygonal boundary (which we call *walls*). A is a robot free to move, and B is an object whose center b needs to be moved from a specific starting position s to a specific final position f. The only way one can move B is by 'pushing' A against it. To simplify the problem, we assume that there is no friction between the discs or between the discs and the walls. We also assume that B has no inertia, and the motion of B is compliant with the walls. A point where two walls meet is called a *corner*. We also denote by $d(x)$ the distance from a point x to the nearest wall.

First, we observe that the direction which B can be moved is restricted when $d(b) < 2r_A + r_B$, where r_A and r_B are the radii of A and B respectively. In Figure 1, the direction that b can be moved is limited by $\overline{bc}$ and $\overline{bd}$. In fact, since $\angle cbd < \pi$, once b is moved to b', it can never be moved back to b again. This shows the *directedness* of the region. Region R is *directed* if for any two points $x, y \in R$ such that b can be moved from position x to position y along a path in R, b can not be moved from y to x along any path that lies in R; otherwise R is *undirected*.

Second, we observe that within certain regions, B can be moved from one position to another position and then be moved back, but the two loci are different in nature. In Figure 2, b can be moved from b to b' along the line $\overline{bb'}$, but the trip back must take a zigzag locus because of the restriction on the position of A[1]. This shows the *asymmetry* of the region. Region R is *asymmetric* if not every locus of the motion of b from x to y is a possible locus of that from y to x, for any two points $x, y \in R$, otherwise R is *symmetric*. Note that if R is *directed* then R must be *asymmetric*, but not *vice versa*.

We call the set of admissible positions of a the *free space* for A, and denote it by F_a. And we define F_b in the same way. For any position x, we denote the circle of radius $(r_A + r_B)$ centered at x by $C(x)$. A room is *reachable* if for any position $x \in F_b$, $C(x) \cap F_a \neq \emptyset$. From here on, we only consider reachable rooms, unless otherwise stated.

The free space F_b can be divided into six types of regions:

- A 2-D region which is both undirected and symmetric is called an α-region.
- A 2-D region which is undirected and asymmetric is called a β-region.
- A 2-D region which is directed is called a γ-region.
- A 1-D region which is undirected and symmetric is called a δ-region.
- A 1-D region which is directed is called an ε-region.
- A point such that b can only be moved into but not from it is called a ζ-region or a *black hole*[2].

[1] We assume that A can be 'lifted' and placed at will.
[2] In an unreachable room, ζ-regions may have dimensions higher than zero.

Depending upon the geometry of the room and discs, there may not exist all the regions listed above. For example, there will be no α-region if for any point $x \in F_b$, $d(x) < 2r_A + r_B$.

Theorem 1. The regions can only be connected in the following ways ($\phi \Longrightarrow \psi$ means B can be pushed from a ϕ-region directly into a ψ-region): $\alpha \Longrightarrow \beta$, $\beta \Longrightarrow \alpha$, $\beta \Longrightarrow \delta$, $\delta \Longrightarrow \beta$, $\beta \Longrightarrow \gamma$, $\gamma \Longrightarrow \delta$, $\gamma \Longrightarrow \varepsilon$, $\delta \Longrightarrow \varepsilon$, $\gamma \Longrightarrow \zeta$, $\delta \Longrightarrow \zeta$, and $\varepsilon \Longrightarrow \zeta$.

A room is *sparse* if for any position $x \in F_b$, $T(x)$ intersects with at most two walls.

We denote the ratio r_A/r_B by ρ.

Theorem 2. A sparse room is reachable if and only if

$$\rho < 1, \text{ or } \rho \geq 1 \text{ and for any corner } \theta \geq 2\arcsin(\frac{\rho - 1}{\rho + 1}).$$

Since we assume no friction and inertia, the locus of b follows a special 'pushing aside' curve (which we call a P-curve) when the direction of pushing keeps constant but is not along $\overline{ab}$. When A moves along $\overline{pq}$ (which we call *major axis*) and the initial position of b is not on $\overline{pq}$, B will be pushed until $\overline{ab} \perp \overline{pq}$ (where its center position b is called an ***end point***).

Theorem 3. If the major axis is the y-axis, and the end point is $(L, 0)$[3], then the P-curve is defined by:

$$y = \sqrt{L^2 - x^2} + L \ln \frac{x}{L + \sqrt{L^2 - x^2}}.$$

Theorem 4. In a sparse room, a corner angle of θ induces a ζ-region if and only if

$$\rho > \frac{1}{3}, \text{ and } 2\arcsin\max\{(\frac{1-\rho}{1+\rho}), 0\} < \theta \leq \arccos(\frac{1-\rho}{1+\rho}).$$

The corner also induces two ε-regions when the equality does not hold.

Now we characterize the boundary between regions.

From definition of the regions and theorem 1, we know that α-regions and β-regions are separated by line segments parallel to the walls at distance $(2r_A + r_B)$, and all the ε-regions (if there are any) are line segments parallel to the walls at distance r_B. If $\rho < 1$, there are no δ-regions, otherwise, there are two types of δ-regions: (a) line segments parallel to the walls at distance r_A (which we call δ_1-regions), and (b) the boundary between β-regions and γ-regions (which we call δ_2-regions).

Theorem 5. For a convex sparse room, the boundary between β-regions and γ-regions consists only of line segments, circular arcs and P-curve segments.

Proof. We first consider the case where $\rho > 1$, *i.e.* $F_b \subseteq F_a$. For any point $x \in F_b$ that is at least $(r_A + r_B)$ away from any vertex on the boundary of F_a, x belongs to a δ-region if $d(x) = r_A$ or $d(x) = r_B$, otherwise, x belongs to a γ-region if $r_B < d(x) < r_A$ and x belongs to a β-region if $r_A < d(x) < 2r_A + r_B$. See Figure 3.

When x is within the distance of $(r_A + r_B)$ from a vertex v on the boundary of F_a, the situation becomes complicated. The boundary between the β-region and γ-region is determined by (i) $C(v)$, (ii) the lines l_1 and l_2 that are parallel to the walls at distance r_A, (iii) the P-curve passing the intersection point between $C(v)$ and l_1 and having l_2 as its major axis, and (iv) the P-curve passing the intersection point between $C(v)$ and l_2 and having l_1 as its major axis. See Figure 4.

It can be shown that when $\theta \leq \pi/4$, the P-curves are outside the circular arc, so the boundary consists of only the circular arc. When $\pi/4 < \theta < \phi$, where ϕ is the angle that makes both P-curves intersect with $C(v)$ at its mid point, the boundary consists of (i) the part of $C(v)$ that lies within its intersection points with the P-curves, and (ii) the P-curve segments. It can be shown that ϕ satisfies the following equation[4]:

$$2(\cos\frac{\phi}{2} - \cos\phi) = \ln(\sin\phi(1 + \cos\frac{\phi}{2})) - \ln(\sin\frac{\phi}{2}(1 + \cos\phi)).$$

[3] According to definition, we have $L = r_A + r_B$.

[4] $\phi \approx 61°53'00''$.

When $\phi \leq \theta < \pi$, The intersecting point of the two P-curves locates on or inside $C(v)$, so the boundary consists of only two P-curve segments. This concludes the case for $\rho > 1$.

When $\rho \leq 1$, the γ-regions can only exist near corners. As in the above case, the boundary between a β-region and a γ-region is determined by the circle $C(u)$ (u is the vertex on the boundary of F_b), the two line segments parallel to the walls at distance r_B, and the two P-curve segments defined similarly. See Figure 5.

When $\rho \leq 1/3$ or $\theta \leq 2\phi(\rho)$, where $\phi(\rho) = \arcsin\frac{1-\rho}{1+\rho}$, or $\theta > \arccos(\frac{1-\rho}{1+\rho})$, there is no γ-region at all; otherwise, the shape of the boundary depends on ρ and θ. If $2\phi(\rho) < \theta \leq \phi(\rho) + \pi/4$, the boundary consists of only the arc on $C(u)$. If $\phi(\rho) + \pi/4 < \theta \leq \psi$, where ψ satisfies

$$2(\cos\frac{\psi}{2} - \cos(\psi - \phi(\rho))) = \ln(\sin(\psi - \phi(\rho))(1 + \cos\frac{\psi}{2})) - \ln(\sin\frac{\psi}{2}(1 + \cos(\psi - \phi(\rho)))),$$

then the boundary consists of both P-curve segments and part of $C(u)$. Finally, the boundary consists of only the two P-curve segments when $\psi < \theta \leq \arccos(\frac{1-\rho}{1+\rho})$. *Q. E. D.*

Theorem 6. In a sparse room, the boundary between β-regions and γ-regions consists of line segments, circular arcs, P-curve segments and ellipsoid arcs.

Proof. In theorem 5, we have considerd the case for corner angle $\theta \leq \pi$. For a corner of $\theta > \pi$, if $\rho \leq 1$, there is no γ-region near the corner; otherwise the boundary between the β-region and the γ-region is determined by segments of the following ellipsoid arcs that are closest the the corner: (a) the locus of the mid point of a line segment of length $2(r_A + r_B)$ when its end points slide along (i) the line parallel to the wall at distance r_A, and (ii) the circular arc of radius r_A centered at the corner, respectively, and (b) the locus of this mid point when the two end points slide along the two lines parallel to the walls at distance r_A. Particularly, when

$$\theta \geq 2\arctan(1+\rho) + 2\arctan(\frac{\rho}{\sqrt{2(1+\rho)}}),$$

the γ-region near the corner splits into two separate parts. See Figure 6. *Q. E. D.*

Corollary 1. A corner of angle θ causes a δ_1-region to connect to a β-region if and only if

$$\rho > 1, \quad \text{and} \quad \theta > 2\arctan(1+\rho) + 2\arctan(\frac{\rho}{\sqrt{2(1+\rho)}}).$$

Theorem 7. In a sparse room with n walls, the number of regions is at most $6n$.

Theorem 8. In a sparse room, the connection between the regions can be determined in time $O(n^{2.8})$, where n is the total number of walls.

Theorem 9. If x and y are two points in a γ-region, and b can be pushed from x to y along a path totally within the region, then the motion can be accomplished by at most two linear movements of A.

However, for motion planning in β-region, the complexity of motion depends on the location of the starting and final position. For example, in Figure 2, the motion from b' back to b takes more steps when b' is closer to the boundary between β-region and γ-region (shown as dash line). When b' approaches this boundary, the number of steps required to move from b' to b approaches infinity.

In conclusion, a class of 2-D compliant motion planning problems is studied. The space under such a model demonstrates some unusual properties not shown in other motion planning problems. We also characterize the regions and the connection among them.

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References

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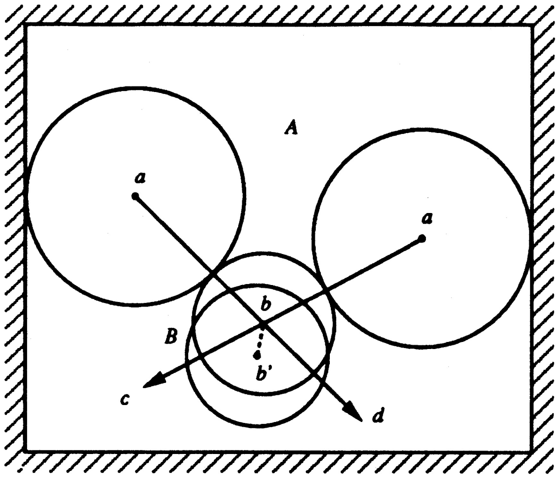

Figure 1.

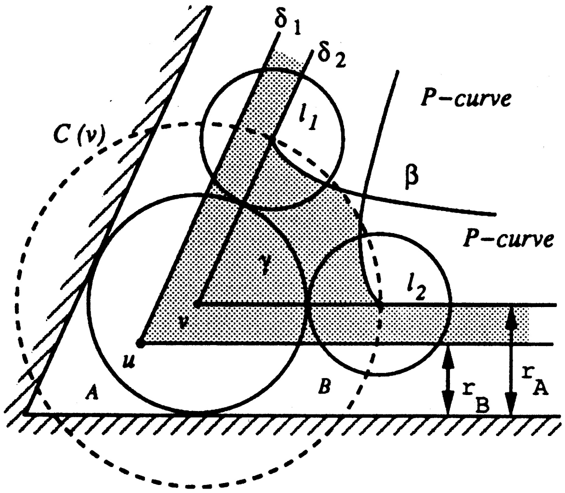

Figure 4.

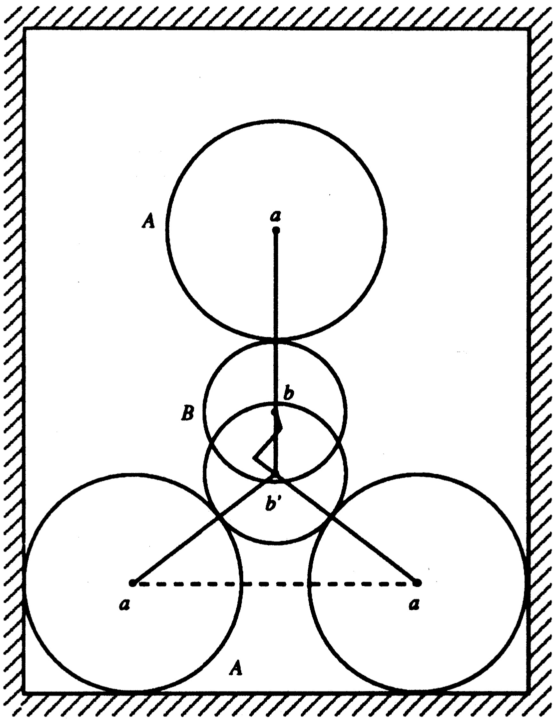

Figure 2.

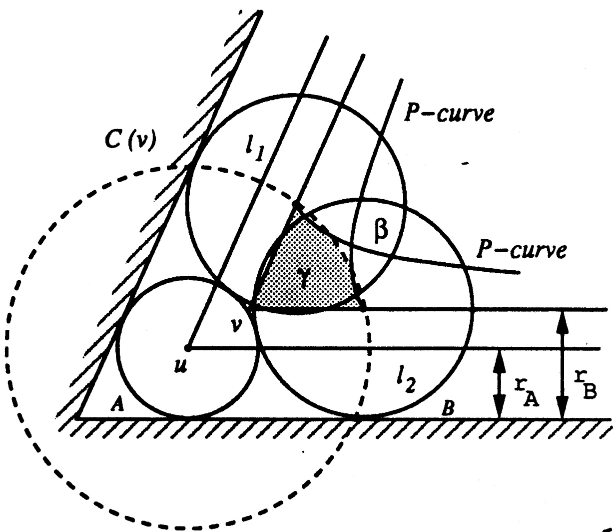

Figure 5.

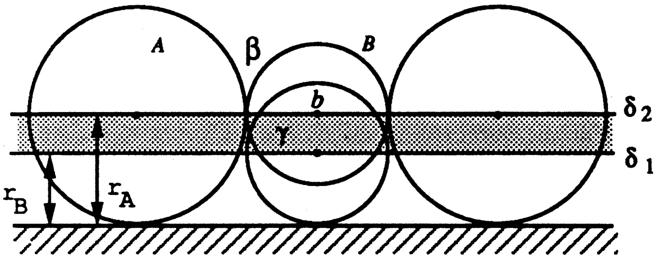

Figure 3.

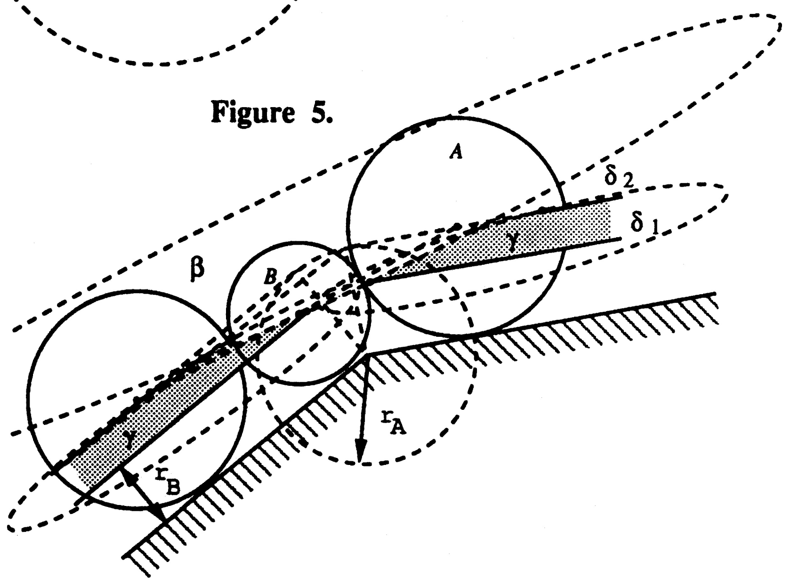

Figure 6.