Turning a Polygon Inside Out

William J. Lenhart
Dept. of Computer Science
Williams College

Sue H. Whitesides
School of Computer Science
McGill University

Extended Abstract

We present a simple necessary and sufficient condition for turning a polygon inside out. The vertices of the polygon, which are labeled, act like rotary joints. The edges are fixed-length links that are allowed to cross over one another as they move. To turn the polygon inside out means to convert it by a continuous motion in the plane to a polygon that is the mirror image (with respect to some arbitrary line in the plane) of the original one. Our simple necessary and sufficient condition is that the lengths of the second and third longest links sum to no more than half the perimeter of the polygon. We then use this condition to solve the following motion planning problem: given two configurations of a chain of links, the two endpoints of which are fixed, determine whether the chain can be moved from one configuration to the other. In fact, this can be determined in linear time. Furthermore, in the event of a yes answer, a sequence of simple motions achieving the reconfiguration can be computed in linear time.

1 Introduction

The problem of reconfiguring chains of links under various conditions has been considered in [1], [2] and [3]. In particular, these papers present polynomial time algorithms for motion planning problems that have an unbounded number of degrees of freedom. While there are general techniques for solving motion planning problems having a bounded number of degrees of freedom in polynomial time, problems having an unbounded number of degrees of freedom are often at least NP-complete. Hence it is interesting to see examples of motion problems that can be solved quickly despite having an unbounded number of degrees of freedom. This paper contributes such an example.

Given a sequence (x, y, z) of three non-collinear points in the plane, we call the sequence a ***left turn*** if it determines a counterclockwise cycle; otherwise we call the sequence a ***right turn***. We say that two sequences (x, y, z) and (x', y', z') have *opposite* orientations if one is a left turn and the other is a right turn; otherwise, we say they have the ***same*** orientation.

A ***chain*** is a sequence of n ***links***, $L_1, ..., L_n$, connected by *joints*. L_i has joints v_{i-1} and v_i and length l_i. Sometimes L_i will be denoted $[v_{i-1}, v_i]$. Each link can rotate freely about its joints. A *configuration* of a chain $L_1, ..., L_n$ is a polygonal curve (possibly self-intersecting) that consists of n consecutive links of lengths $l_1, ..., l_n$, respectively. A *closed chain* is chain such that v_0 and v_n are the same joint. Hence a configuration of a closed chain is just a closed polygonal curve. An *arm* is a chain in which a fixed location in the plane has been associated with v_0.

We use the term *linkage*, often denoted L, to refer to a chain, a closed chain or an arm.

Definition 1.1 *Two configurations of a linkage L are* equivalent *if one configuration can be continuously transformed to the other in such a way that the lengths of the links remain fixed throughout the motion.*

In mathematical terms, this means that there is a homotopy between the two curves having the property that the link lengths are preserved *throughout* the homotopy. If two configurations of L are equivalent, we say that L can be *moved* from the one configuration to the other.

Definition 1.2 *A configuration of a closed chain L is* invertible *if it is equivalent to its mirror image (with respect to some arbitrary line).*

Since a simple polygon can be viewed as a configuration of a closed chain, we will refer to a polygon as *invertible* if it is equivalent to some mirror image of itself. Figure 1 shows a polygon that is invertible.

Key idea: Let $L_i = [v_{i-1}, v_i]$ be a link of a closed chain L. Suppose that some configuration of L can be moved to its mirror image with respect to some arbitrary line. Let v_j be any joint of L that is not collinear with L_i in the original configuration. Then at some moment during the motion, v_j lies on the the line through L_i at that moment. This is because the orientation of (v_j, v_{i-1}, v_i) in the original configuration of L differs from its orientation in any mirror image of that configuration. By determining what conditions on the lengths of the links allow each v_j to become collinear with each L_i, we can establish a necessary condition for invertibility; as it happens, this condition is also sufficient.

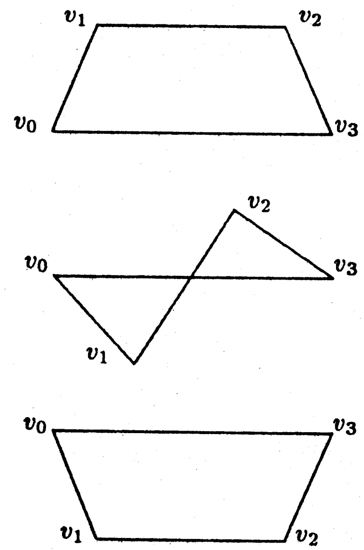

Figure 1: A polygon being inverted. $[v_0, v_1]$ rotates c.w. about v_0 as $[v_2, v_3]$ rotates first c.w., then c.c.w. about v_3.

Definition 1.3 *Given a configuration of a closed chain L, a joint v of L is said to be* extensible *if the configuration is equivalent to a configuration in which the angle between the two links adjacent to v is $180°$. (When the angle of a joint equals $180°$, we say it is* extended.*) Similarly, v is* collapsible *if the configuration is equivalent to one in which the angle is $0°$. (If a joint has an angle of $0°$ we call it* collapsed.*)*

2 Inverting a Polygon

One of our main results is the following:

Theorem 2.1 *Given a configuration of a closed chain L, the sum of whose link lengths equals m, the configuration can be inverted iff* *the lengths of its second and third longest links sum to no more than $m/2$.*

One consequence of this theorem is that invertibility is seen not to be a property of a particular configuration of a closed chain, but rather a property of the chain itself. That is, if some configuration of L can be inverted, then *every* configuration of L can be inverted.

Lemma 2.1 *Any configuration of a closed chain that forms a simple polygon is equivalent to some configuration that forms a convex polygon.*

This lemma has the following consequence. For a given closed chain L, either all its configurations are equivalent, or they partition into two class. One class includes the convex polygons with clockwise orientation obtainable from L (vertices are visited in order $v_0, v_1, \ldots$ to determine the orientation); the other class contains the convex polygons with the opposite orientation.

A configuration of a linkage is said to be *flat* if all of its links are collinear.

Lemma 2.2 *If a closed chain L admits a flat configuration, then some configuration of L in the form of a convex polygon is invertible.*

Another useful lemma is the following:

Lemma 2.3 *If a configuration of a closed chain L contains a joint that can be collapsed but not extended, then that configuration is invertible.*

Sketch of the proof of Theorem 2.1: Assume in what follows that the chain contains at least four links. First, we establish the necessity of the condition by showing that if the condition is violated, then some joint cannot be positioned on the line determined by some link, which we know to be necessary by the

Key idea. Since the chain has at least four links, by suitably choosing which joint to label v_0, we can assume that L_i, L_j and L_k are the three longest links, that L_j and L_k do not have a joint in common, and that $i < j < k$. (L_i, L_j, and L_k do not necessarily appear in order of decreasing length.) See Figure 2.

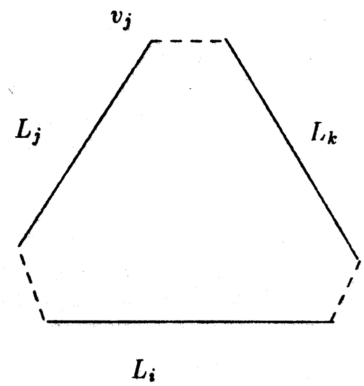

Figure 2: A polygon with longest links L_i, L_j and L_k.

If the condition is violated, we have

$$\begin{aligned} l_i + l_j &> m/2 \\ l_i + l_k &> m/2 \\ l_j + l_k &> m/2 \end{aligned} \tag{1}$$

Partition L into three chains as follows: the link $L_i = [v_{i-1}, v_i]$, the 'left chain' $(v_i, v_{i+1}, \ldots, v_j)$, and the 'right chain' $(v_{i-1}, v_{i-2}, \ldots, v_j)$. Also, let s_j be the total length of the left chain not including link L_j and let s_k be the total length of the right chain not including link L_k. Thus $m = l_i + l_j + s_j + l_k + s_k$.

For v_j to lie on the line through L_i, it can be shown that at least one of the following inequalities must be satisfied (these inequalities correspond to v_j being on, to the left of, or to the right of L_i, respectively).

$$\begin{aligned} (l_j - s_j) + (l_k - s_k) &\leq l_i \\ (l_j - s_j) + l_i &\leq (l_k + s_k) \quad (2) \\ (l_k - s_k) + l_i &\leq (l_j + s_j) \end{aligned}$$

However, each of these inequalities contradicts one of the three inequalities in (1).

Now we give the idea behind the proof of the sufficiency of the condition. First, we show that if there exists a joint v_i that neither extends nor collapses then we can find three links (two of which are L_i and L_{i+1}) such that each pair of links has total length greater than half of the perimeter of L. This clearly violates the condition. We then prove by induction on the number of links in L that the condition is sufficient by demonstrating that either the configuration of L has a joint that collapses but does not extend (in which case we are done by Lemma 2.3), or the configuration of L is equivalent to a configuration in which one of the joints of the longest link is extended. This new configuration can be viewed as a closed chain having one fewer link (the extended joint is 'frozen', making a new longest link), and the condition of the theorem still holds.

3 Application

We give an application to motion planning. From [1] we use:

Definition 3.1 *A* simple motion *of a linkage is a continuous motion such that*

1. *At most four joint angles are changing. (If the linkage is an arm, then one of the angles may be the angle formed by l_1 and some line through v_0.)*
2. *Each angle is changing monotonically.*

The main result is:

Theorem 3.1 *Given two configurations of a chain C that has both endpoints fixed at particular points in the plane, it can be determined in linear time whether the two configurations are equivalent. Moreover, if the configurations are equivalent, it is possible to compute in linear time a sequence of simple motions that moves one of the configurations to the other.*

References

[1] John Hopcroft, Deborah Joseph, and Sue Whitesides. On the movement of robot arms in 2-dimensional bounded regions. *SIAM J. Comput.*, 14, No.2:315–333, 1985.

[2] V. Kantabutra and S. R. Kosaraju. New algorithms for multilink robot arms. *J. Comput. Sys. Sci.*, February, 1986.

[3] Vitit Kantabutra. Motions of a short-linked robot in a square. *Discrete and Computational Geometry*, Accepted for Publication.