Minimal circumscribing simplices

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Introduction

In recent years considerable attention has been devoted to the problem of approximating geometric objects by simpler objects. In computer graphics one approximates polyhedral models by bounding balls or bounding boxes to obtain fast rendering algorithms. Similar techniques are applied in the context of motion planning to detect collisions between moving obstacles. In the plane very efficient algorithms have been developed for circumscribing or inscribing a given polyon with convex k-gons of minimal or maximal area, respectively.

In [2] an $O(n \log^2 n)$ algorithm is obtained for computing a minimum area circumscribing triangle of a convex n-gon. This time bound has been improved to $O(n)$ in [5]. A unified approach to both problems is contained in [3]. The algorithms in [2] and [3] determine a large class of circumscribing triangles that are *locally minimal* in the sense that any circumscribing triangle that is a sufficiently small perturbation of the given triangle does not have smaller area. We shall use the term locally minimal exclusively in this sense.

In this paper we analyze simplices with *locally minimal volume* circumscribing a convex polytope in $\mathbb{R}^d$. This constitutes a first step towards the development of an algorithm that computes a circumscribing simplex with minimal volume.

A necessary condition for local minimality is that all centroids of facets lie on the circumscribed polyhedron. However, if these are the only constraints the volume is not a local minimum. Therefore we consider additional constraints like fixing the supporting hyperplane of some facets, corresponding to the situation in which the polyhedron has a facet that is flush with a facet of the simplex circumscribing it. For polytopes in $\mathbb{R}^d$ we give a complete classification of locally minimal circumscribing simplices each of whose facets has a contact with the polytope either of maximal dimension ($d-1$, flush facet) or of minimal dimension (0, contact at facet centroid).

It turns out that there can be locally minimal circumscribing simplices whose facets have contacts of neither minimal nor maximal dimension. We give a complete classification only in for polytopes in 3-space.

Crucial in our approach are *barycentric coordinates.* Especially when dealing with volumes they turn out to be very convenient. We first introduce these coordinates and state some properties relevant for our problem. Then we consider necessary and, in 3-space, sufficient conditions for local minimality of the volume function.

Barycentric coordinates

Fix $d+1$ points $p^1, \cdots, p^{d+1}$ in $\mathbb{R}^d$ that are affinely independent. These points are the vertices of a d-dimensional simplex Σ. For each point x in $\mathbb{R}^d$ the *barycentric coordinates* $(\xi_1, \cdots, \xi_{d+1})$ are the weights that have to be put in the points $p^1, \cdots, p^{d+1}$, respectively, in order to make x their center of mass. Obviously barycentric coordinates are homogeneous. In other words, $\xi_1, \cdots, \xi_{d+1}$ are determined *up to a scalar factor* by the condition

$$x = \sum_{m=1}^{d+1} \xi_m p^m \Big/ \sum_{m=1}^{d+1} \xi_m \qquad (1)$$

These coordinates can be made unique by imposing the constraint

$$\sum_{m=1}^{d+1} \xi_m = 1 \qquad (2)$$

The numbers ξ_m, $1 \leq m \leq d+1$, satisfying (1) and (2) will be called *normalized barycentric coordinates* of the point x.

In particular the point p^m has normalized barycentric coordinates δ_n^m, $1 \leq n \leq d+1$. (Here δ_n^m is the Kronecker delta-symbol.) The d-simplex Σ_p with vertices $p^1, \cdots, p^{d+1}$ is called the *simplex of reference* for the barycentric coordinates just introduced.

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Consider a simplex Σ_q with vertices $q^1, \cdots, q^{d+1}$. The normalized barycentric coordinates of q^m (w.r. to Σ_p) are denoted by $(q_1^m, \cdots, q_{d+1}^m)$. The following basic properties of barycentric coordinates are the basic tools in our approach.

BC_1. Let Q be the $(d+1)\times(d+1)$-matrix whose m-th row consists of the normalized barycentric coordinates of q^m, i.e. $Q = (q_n^m)_{1\leq m,n\leq d+1}$. Then

$$\frac{\mathrm{Vol}(\Sigma_q)}{\mathrm{Vol}(\Sigma_p)} = |\det Q|,$$

BC_2. If the vertices $p^1, \cdots, p^{d+1}$ are the *centroids* of the facets, i.e. $(d-1)$-dimensional faces, $f^1, \cdots, f^{d+1}$, respectively, then the normalized barycentric coordinates of q^m with respect to the simplex of reference Σ_p are $(q_1^m, \cdots, q_{d+1}^m)$, where

$$q_n^m = \begin{cases} 1-d, & \text{if } m = n, \\ 1, & \text{if } m \neq n. \end{cases}$$

BC_3. Suppose a^m has barycentric coordinates $a_1^m, \cdots, a_{d+1}^m$, $1 \leq m \leq d+1$, then points $a^1, \cdots, a^{d+1}$ are *affinely independent (in general position)* iff.

$$\begin{vmatrix} a_1^1 & \cdots & a_{d+1}^1 \\ \vdots & & \vdots \\ a_1^{d+1} & \cdots & a_{d+1}^{d+1} \end{vmatrix} \neq 0$$

In the sequel we consider simplices Σ obtained by perturbing a simplex Σ_0 within the class of simplices whose m-th face is constrained to contain the centroid p^m of the m-th face of Σ_0. We use normalized barycentric coordinates whose simplex of reference has vertices $p^1, \cdots, p^{d+1}$. In particular p^m has barycentric coordinates δ_n^m, $1 \leq n \leq d+1$, and therefore vertices of Σ_0 have barycentric coordinates $1-\delta_n^m d$, $1 \leq n \leq d+1$, cf. property BC_2.

Let $q^1, \cdots, q^{d+1}$ be the vertices of Σ, and let f^m be its m-th face (opposite q^m). Then the normalized barycentric coordinates of q^m are of the form $1-\delta_n^m d+\xi_n^m$, $1 \leq n \leq d+1$, where $(\xi_1^m, \cdots, \xi_{d+1}^m)$ ranges over a neighborhood of $(0, \cdots, 0)$ in $I\!R^{d+1}$, subject to the *normalization constraint* $\sum_{n=1}^{d+1} \xi_n^m = 0$.

Constrained volume minimization

The following result gives a necessary condition for the situation in which a circumscribing simplex has locally minimal volume. In the full version of this paper we provide a proof using barycentric coordinates.

Theorem 1 (V. Klee[1]) *Consider a simplex Σ with one facet constrained by a $(d-2)$ flat that does not contain the centroid of this facet. Then an arbitrarily small perturbation of this facet that respects the constraint yields a simplex whose volume is strictly smaller than the volume of Σ provided the centroid of the unperturbed facet lies outside the new simplex.*

Corollary 2 *If a circumscribing simplex Σ for the polytope P has locally minimal volume then the centroids of the facets of Σ lie on P. (In this case Σ is called a critical simplex.)*

We shall consider circumscribing simplices that are perturbations of a critical simplex and also satisfy certain constraints concerning the contact with the polytope they circumscribe. The mildest constraint is that each facet contains a fixed point (which is a facet centroid of the unperturbed simplex). We call this a constraint of type V, since a polytope can only touch a facet in a single point if this point is a vertex of the polytope. The most severe constraint fixes the supporting hyperplane of a facet of the simplex. This constraint is of type F. It corresponds to the situation in which a facet of the circumscribed polytope is flush with a facet of the simplex.

We shall only consider contacts of intermediate dimension in the 3-dimensional situation. This class of constraints is of type E, corresponding to the situation in which an edge of the polytope is flush with a face of the circumscribing simplex (a tetrahedron in this case).

Using this notation the constraint V^aF^b, with $a+b = d+1$, corresponds to a situation in which exactly b facets of the simplex have fixed supporting hyperplanes, and the a remaining facets are merely constrained by the centroid of Σ. In three dimensions we also consider constraints of type $V^aE^bF^c$, with $a+b+c = d+1$. The meaning of this notation is obvious. For our purposes it is irrelevant which facets satisfy a constraint of type F, E or V, so we don't express this in our notation.

First we consider perturbations of a critical simplex satisfying a constraint of type V^aF^b with $a+b = d+1$, for various values of a.

Theorem 3 *Consider a critical simplex Σ_0.*

1. *Among all simplices satisfying a constraint of type VF^d the volume of Σ_0 is strictly locally minimal.*
2. *Among all simplices satisfying a constraint of type V^2F^{d-1} the volume of Σ_0 is locally minimal. The minimum is not isolated: the set of locally minimal volume simplices near Σ_0 is a $(d-1)$-dimensional manifold.*
3. *Among all simplices satisfying a constraint of type V^kF^{d+1-k}, with $k \geq 3$, the volume of Σ_0 is* **not** *locally minimal.*

Proof. 1. As indicated above the perturbation parameters are ξ_n^m, measuring the deviation of the n-th barycentric coordinate of q^m from the value $1-\delta_n^m$ in the unperturbed situation.

We fix the supporting planes of $f^2, \cdots, f^{d+1}$. Then for $1 \leq m \leq d+1$

$$\xi_n^m = 0, \text{ for } 2 \leq n \leq d+1 \text{ and } n \neq m.$$

Therefore the normalization constraint $\sum_{n=1}^{m} \xi_n^m = 0$ implies $\xi_1^1 = 0$. Using property BC_1 we easily derive

$$\mathrm{Vol}(\xi) = \frac{\mathrm{Vol}(0)}{d^d}(d-\xi_2^2)(d-\xi_3^3)\cdots(d-\xi_{d+1}^{d+1}). \quad (3)$$

Since facet f^1 is constrained to contain $p^1 = (1,0,\cdots,0)$ we use property BC_3 to obtain

$$0 = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1+\xi_1^2 & 1-d+\xi_2^2 & \cdots & 1 \\ 1+\xi_1^3 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1+\xi_1^{d+1} & 1 & \cdots & 1-d+\xi_{d+1}^{d+1} \end{vmatrix}$$

$$= (1+\sum_{m=2}^{d+1}\frac{1}{\xi_m^m-d})(d-\xi_2^2)\cdots(d-\xi_{d+1}^{d+1}). \quad (4)$$

In view of (3) and (4) $1/\mathrm{Vol}(\xi)$ is maximal for ξ near 0 iff. for $1 \leq m \leq d+1$

$$\frac{1}{\xi_m^m - d} = -\frac{1}{d},$$

i.e. if $\xi_m^m = 0$. But then also $\xi_1^m = 0$. Therefore $\xi = 0$ is an *isolated* minimum of the volume function $\mathrm{Vol}(\xi)$.

2. Similar to the first part of the proof we derive

$$\mathrm{Vol}(\xi) = \frac{\mathrm{Vol}(0)}{d^{d-1}}(d-\xi_1^1-\xi_2^2)(d-\xi_3^3)\cdots(d-\xi_{d+1}^{d+1}),$$

and

$$\frac{1}{d-\xi_1^1-\xi_2^2} + \sum_{m=3}^{d+1}\frac{1}{d-\xi_m^m} = 1.$$

Therefore $\mathrm{Vol}(\xi)$ has a local minimum at $\xi = 0$, but the minimum is not isolated. It is obvious from the equations above that the set $\{\xi \mid \mathrm{Vol}(\xi) = \mathrm{Vol}(0)\}$ is a $(d-1)$-dimensional manifold.

3. The last part of the proof will be given below. □

The quadratic part of the volume function

To determine the nature of the singular point $\xi = 0$ of the volume function V in the presence of various types of constraints we need to know the second order terms of this function V. (Obviously there are no linear terms.) In particular V has a local minimum if this second order part is positive definite.

For convenience we scale the normalized barycentric coordinates by introducing new variables η_n^m defined for $1 \leq m,n \leq d+1$ and $m \neq n$ by:

$$\eta_n^m = \frac{\xi_n^m}{\xi_m^m - d}.$$

Note that the coordinates η_n^m can range over a full neighborhood of the origin $\eta_n^m = 0$, $1 \leq m,n \leq d+1$ and $m \neq n$, in $I\!R^{d^2}$, i.e. they don't have to satisfy any constraint.

Theorem 4 *If the faces of the simplex satisfy the default constraint, i.e. p^k is contained in the hyperplane supporting face f^k, $1 \leq k \leq d+1$, then*

$$V(\eta) = V(0)(1 + \sum_{\substack{m,n=1 \\ m<n}}^{d+1} \eta_n^m \eta_m^n + \frac{1}{2}\sum_{m=1}^{d+1}((\zeta^m)^2) + O(|\eta|^3)$$

where

$$\zeta^m \stackrel{\text{def}}{=} \sum_{\substack{n=1 \\ n\neq m}}^{d+1} \eta_n^m$$

Proof of theorem 3 (continued): We shall give an example of a simplex Σ, obtained by an arbitrarily small perturbation of the critical simplex Σ_0, satisfying a constraint of type $V^k F^{d+1-k}$, $k \geq 3$, such that $\mathrm{Vol}(\Sigma) < \mathrm{Vol}(\Sigma_0)$.

Let faces f^1, f^2 and f^3 satisfy a constraint of type V. We may fix the supporting hyperplanes of the remaining faces, viz. for $1 \leq m \leq d+1$ we take $\eta_n^m = 0$, for $4 \leq n \leq d+1$ and $n \neq m$. We restrict the number of degrees of freedom of f^1, f^2 and f^3 to 1 by fixing $(d-2)$-flats that are the affine hulls of $\{p^m, q^4, \cdots, q^{d+1}\}$, for $1 \leq m \leq 3$. It is easy to check that these additional constraints fix vertices q^m, $4 \leq m \leq d+1$, i.e. $\eta_n^m = 0$ for $4 \leq m \leq d+1$, $1 \leq n \leq n$ and $n \neq m$. We are now left with 3 degrees of freedom. Taking $\eta_2^1 = x$, $\eta_3^2 = x$ and $\eta_1^3 = x$ property BC_3 can be used to prove $\eta_3^1 = -x + O(x^2)$, $\eta_1^2 = -x + O(x^2)$ and $\eta_2^3 = -x + O(x^2)$, where x is a real variable ranging over a neighborhood of 0 in $I\!R$. Then by definition we have $\zeta^m = O(x^2)$ for all m. So the volume V is a function of x satisfying (see theorem 4)

$$V(x) = V(0)(1-3x^2) + O(x^3).$$

Hence $V(x) < V(0)$ for sufficiently small $x \neq 0$. In other words, the volume of Σ is *not locally minimal*. □

The 3-dimensional case

The results of the previous section can be applied to obtain a characterization of local minima in the 3-dimensional situation, in particular to determine when the volume function has a local minimum if we add several constraints of type E.

Theorem 5 *Let Σ_0 be a critival simplex circumscribing a polyhedron P. If the volume of Σ_0 is locally minimal then either P and Σ_0 have at least two flush faces, or their contact is of type E^3F or E^4.*

The proof of this theorem involves heavy calculations using the barycentric machinery developed above, especially theorem 4. Using these calculations it is easy to construct examples of locally minimal circumscribing simplices of any of the types referred to in the theorem.

An intermediate step in this proof is a geometric characterization of local minima of type E^3F. Since it might have some interest of its own, we state the result below.

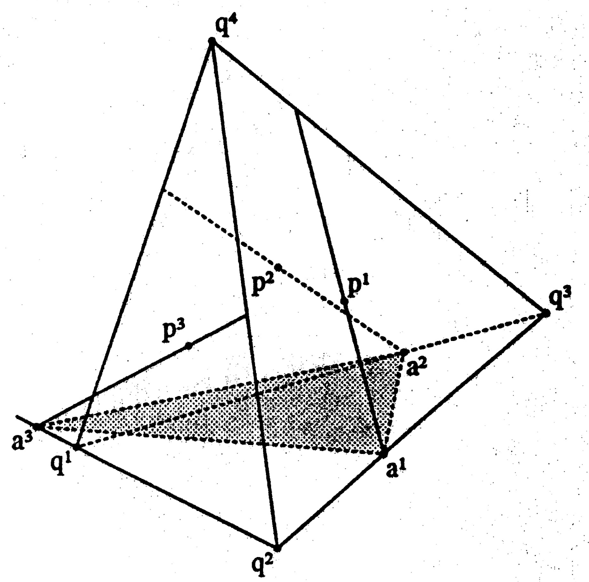

Figure 1: A constraint of type E^3F

Let q^m, $1 \leq m \leq 4$, be the vertices of a simplex, see Figure 1. Each of the faces f^1, f^2 and f^3 contains a line through its centroid p^1, p^2 and p^3, respectively. These lines intersect the supporting plane of face f^4 in points a^1, a^2 and a^3, respectively. We consider perturbations of this simplex in which the supporting plane of face f^4 is fixed (constraint F), and in which each remaining face is perturbed subject to the condition that its supporting planes should contain the line through the centroid in the unperturbed situation (constraint E^3).

Lemma 1 *Consider tetrahedra satisfying a constraint of type E^3F as described previously. For such tetrahedra let Δ be the ratio of the signed area of triangle $a^1a^2a^3$, and the area of the face whose supporting plane is fixed.*

Among all tetrahedra that are perturbations of a critical tetrahedron T and satisfy a constraint of type E^3F the volume of T is strictly locally minimal iff

$$\Delta > \frac{1}{3}.$$

Concluding remarks

Further research is planned concerning the construction of a globally minimal circumscribing simplex for a convex polytope P, first of all in 3-space. In [4][1] all local minima with at least two flush faces are determined in $O(N^3 \log N)$ time, where N is the number of vertices of P. We also have an $O(N^3 \log N)$ method to determine all local minima of type E^3F. The problem of finding all local minima of type E^4 in $o(N^4)$ time is still open.

References

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[1] In fact the latter paper served as a starting point for our work. We are indebted to Joe O'Rourke for this work, and for his helpful criticism.