On the perimeter of a point set in the plane

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Abstract

Let $\{c_0,\dots,c_n\},\{c'_0,\dots,c'_n\}$ be two point sets in the plane satisfying $|c_i-c_j|\le|c'_i-c'_j|$ for all i and j. By a theorem of Sudakov and Alexander, the perimeter of the convex hull of $\{c_0,\dots,c_n\}$ does not exceed the perimeter of the convex hull of $\{c'_0,\dots,c'_n\}$. We give a simple proof of this result and establish a similar theorem in the case when the Euclidean distance is replaced by the maximum norm. We point out the close relationship between these questions and a longstanding open problem due to Thue Poulsen, Kneser and Hadwiger.

1 The union of balls.

More than 35 years ago E. Thue Poulsen [TP], M. Kneser [K] and H. Hadwiger [H] proposed the following conjecture which has attracted the interest of many geometers but is still open. Let $\{C_0,\dots,C_n\},\{C'_0,\dots,C'_n\}$ be two collections of disks of radius r in the plane. Let c_i and c'_i denote the center of C_i and C'_i, respectively, and assume that

$$|c_i-c_j|\le|c'_i-c'_j| \text{ for all } i \text{ and } j.$$

Then

$$\text{Area}\Big(\bigcup_{i=0}^{n} C_i\Big)\le\text{Area}\Big(\bigcup_{i=0}^{n} C'_i\Big). \qquad (1)$$

W. Habicht (see [K]) and B. Bollobás [Bo] settled the special case when the system $\{C_0,\dots,C_n\}$ can be continuously transformed into $\{C'_0,\dots,C'_n\}$ so that during the transformation the mutual distances between the centers do not decrease.

Assume now that (1) is true, and let the radii r of the disks tend to infinity. It is easy to see that

$$\text{Area}\Big(\bigcup_{i=0}^{n} C_i\Big)=r^2\pi+r\,\text{Per conv}\{c_0,\dots,c_n\}$$

$$+\text{Area conv}\{c_0,\dots,c_n\}+O\Big(\frac{1}{r}\Big),$$

where Per and conv stand for the perimeter and the convex hull, respectively. Similarly,

$$\text{Area}\Big(\bigcup_{i=0}^{n} C'_i\Big)=r^2\pi+r\,\text{Per conv}\{c'_0,\dots,c'_n\}$$

$$+\text{Area conv}\{c'_0,\dots,c'_n\}+O\Big(\frac{1}{r}\Big),$$

Now (1) immediately yields

$$\text{Per conv}\{c_0,\dots,c_n\}\le\text{Per conv}\{c'_0,\dots,c'_n\}. \qquad (2)$$

The aim of the present note is to give a simple elementary proof of this weaker assertion, which was first established by Sudakov ([S]) and rediscovered by Alexander ([A]). Our approach is similar to the one followed in Alexander's paper, but it avoids using Schläfli's formula. Both proofs are based on a simple property of simplices (Lemma 1 in the next section).

Theorem 1 *Let $\{c_0,\dots,c_n\},\{c'_0,\dots,c'_n\}$ be two point sets in the plane satisfying $|c_i-c_j|\le|c'_i-c'_j|$ for all i and j. Then the perimeter of the convex hull of $\{c_0,\dots,c_n\}$ does not exceed the perimeter of the convex hull of $\{c'_0,\dots,c'_n\}$.*

Let us recall first some basic notions and results from the theory of convex bodies. Let B^n denote the n-dimensional unit ball, and let S^{n-1} be the boundary of B^n. For any convex set $K\subseteq\mathbf{R}^n$, let $K+rB^n$ denote the *parallel body* of K with radius r, i.e., the set of all points of the space whose distance from at least one element of K is at most r. Let Vol_n stand for the n-dimensional volume. It is well-known (see e.g. Bonnesen–Fenchel [BF], Busemann [Bu], Leichtweiss [L]) that $\text{Vol}_n(K+rB^n)$ can be expressed as a polynomial of degree n in r,

$$\text{Vol}_n(K+rB^n)=W_0(K)+\tbinom{n}{1}W_1(K)r \qquad (3)$$

$$+\tbinom{n}{2}W_2(K)r^2+\ldots+W_n(K)r^n.$$

The coefficient $W_m(K)$ is called the m-th mean projection measure (Quermassintegral) of K. $W_0(K)=\text{Vol}_n(K)$, $\binom{n}{1}W_1(K)$ is the surface area of K, $W_n(K)=\text{Vol}_nB^n=\kappa_n$. In general, apart from a factor depending only on n, $W_m(K)$ is the average of $\text{Vol}_{n-m}K(F)$ over all $(n-m)$-dimensional subspaces $F\subseteq\mathbf{R}^n$, where $K(F)$ denotes the orthogonal projection of K on F. In particular,

$$W_{n-1}(K)=\frac{1}{2n}\int_{S^{n-1}}\text{width}_n(K,x)\,dx, \qquad (4)$$

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where $\text{width}_n(K,x) = Vol_1 K(x)$ is the distance between the two supporting hyperplanes of K perpendicular to the unit vector $x \in S^{n-1}$.

Let us turn now to the proof of Theorem 1. Imagine that the plane is embedded in $\mathbf{R}^n$. Put $K = \text{conv}\{c_0, \ldots, c_n\}$, $K' = \text{conv}\{c'_0, \ldots, c'_n\}$. Furthermore, let B_i and B'_i denote the n-dimensional balls of radius r centred at c_i and c'_i respectively. Obviously, $\bigcup_{i=0}^n B_i \subseteq K + rB^n$.

Claim 1 *Let* $\text{diam}\, K = \max_{0 \le i,j \le n} |c_i - c_j|$. *If* $r > \text{diam}\, K$, *then*

$$K + \Big(r - \frac{(\text{diam}\, K)^2}{r}\Big)B^n \subseteq \bigcup_{i=0}^n B_i.$$

Proof: Let $x \in K + (r - \frac{(\text{diam}\, K)^2}{r})B^n$, i.e, there exists $k \in K$ such that $|x - k| \le r - \frac{(\text{diam}\, K)^2}{r}$. If x is not a vertex of K, then one can choose c_i so that $\angle xkc_i \le \frac{\pi}{2}$. But then

$$|x - c_i| \le \sqrt{|x-k|^2 + |k - c_i|^2}$$

$$\le \sqrt{\Big(r - \frac{(\text{diam}\, K)^2}{r}\Big)^2 + (\text{diam}\, K)^2} \le r$$

provided that $r \ge \text{diam}\, K$. Hence, $x \in B_i$. □

Claim 2 *Let B_i and B'_i denote the n-dimensional balls of radius r centred at c_i and c'_i respectively. Then,*

$$\text{Vol}_n(\bigcup_{i=0}^n B_i) \le \text{Vol}_n(\bigcup_{i=0}^n B'_i).$$

Now we show how Theorem 1 can be deduced from these observations, and postpone the proof of Claim 2 until the next section.

Combining Claims 1 and 2, we obtain that

$$\text{Vol}_n\Big(K + (r - \frac{(\text{diam}\, K)^2}{r})B^n\Big) \le \text{Vol}_n(K' + rB^n).$$

Substituting (3), this implies

$$W_n(K)\Big(r - \frac{(\text{diam}\, K)^2}{r}\Big)^n$$

$$+ W_{n-1}(K)\Big(r - \frac{(\text{diam}\, K)^2}{r}\Big)^{n-1} + O(r^{n-2})$$

$$\le W_n(K')r^n + W_{n-1}(K')r^{n-1} + O(r^{n-2}).$$

Using the fact that $W_n(K) = W_n(K') = \kappa_n$, and taking the limits as r tends to infinity, we get

$$W_{n-1}(K) \le W_{n-1}(K'). \tag{5}$$

However, K is a *planar* convex set, so (4) can be rewritten as

$$W_{n-1}(K)$$

$$= \frac{1}{2n}\int_{S^1}\int_0^{+\frac{\pi}{2}}\int_{(\sin\phi)S^{n-3}} \text{width}_2(K,y)\cos^2\phi \, dz d\phi dy$$

$$= \frac{(n-2)\kappa_{n-2}}{2n}\Big(\int_{S^1} \text{width}_2(K,y)\,dy\Big)$$

$$\Big(\int_0^{+\frac{\pi}{2}} \sin^{n-3}\phi\cos^2\phi\, d\phi\Big).$$

Similarly,

$$W_{n-1}(K') = \frac{(n-2)\kappa_{n-2}}{2n}\Big(\int_{S^1} \text{width}_2(K',y)\,dy\Big)$$

$$\Big(\int_0^{+\frac{\pi}{2}} \sin^{n-3}\phi\cos^2\phi\, d\phi\Big).$$

Hence, it follows immediately from (5) that

$$\text{Per}\, K = \frac{1}{2}\int_{S^1} \text{width}_2(K,y)\,dy$$

$$\le \frac{1}{2}\int_{S^1} \text{width}_2(K',y)\,dy = \text{Per}\, K',$$

completing the proof of Theorem 1.

2 Proof of Claim 2.

The proof consists of two steps. The first one is a slight generalization of a simple fact which was also used by M. Gromov [G] to verify a special case of the following 'dual' counterpart of the Hadwiger-Kneser-Thue Poulsen conjecture: Let $\{B_0, B_1, \ldots, B_m\}$, $\{B'_0, B'_1, \ldots, B'_m\}$ be two sets of balls in n-dimensional space, and let c_i and c'_i denote the center of B_i and B'_i, respectively. If

$$|c_i - c_j| \le |c'_i - c'_j| \text{ for all } i \text{ and } j,$$

then

$$\text{Vol}_n(\bigcap_{i=0}^m B_i) \ge \text{Vol}_n(\bigcap_{i=0}^m B'_i). \tag{6}$$

Gromov proved this result when the number of balls does not exceed $n+1$, i.e., $m \le n$.

Lemma 1 *Let $K = \text{conv}\{c_0, \ldots, c_n\}$ and $K' = \text{conv}\{c'_0, \ldots, c'_n\}$ be two non-degenerate simplices in n-dimensional space, and assume that $|c_i - c_j| \le |c'_i - c'_j|$ for all i and j.*

Then K can be continuously transformed into a congruent copy of K' in a finite number of steps so that

1. *in each step we move only one vertex,*
2. *the motion of this vertex is smooth,*
3. *the edgelengths of the simplex never decrease.*

Proof: Omitted in this abstract. □

Lemma 2 *Let $\{B_0, B_1, \ldots, B_m\}$ be a collection of n-dimensional balls of arbitrary radii in $\mathbf{R}^n$, and let c_i denote the center of B_i. Let $\vec{v}$ be a vector with the property that translating c_0 along $\vec{v}$, the distance between c_0 and any other c_i does not decrease. Then $\mathrm{Vol}_n(\bigcup_{i=0}^m B_i)$ does not decrease during this translation.*

Proof: Omitted in this abstract. □

Now we are in the position to prove Claim 2. Let us perturb slightly the point sets $c_0, \ldots, c_n, c'_0, \ldots, c'_n \subseteq \mathbf{R}^n$ to make them full dimensional, without violating the conditions

$$|c_i - c_j| \le |c'_i - c'_j| \text{ for all } i \text{ and } j.$$

Let us assume that this perturbation does not change $\mathrm{Vol}_n(\bigcup_{i=0}^n B_i)$ and $\mathrm{Vol}_n(\bigcup_{i=0}^n B'_i)$ by more than ϵ.

Lemmata 1 and 2 now imply

$$\mathrm{Vol}_n(\bigcup_{i=0}^n B_i) - \epsilon \le \mathrm{Vol}_n(\bigcup_{i=0}^n B'_i) + \epsilon$$

for any $\epsilon > 0$, and Claim 2 follows.

3 Perimeter in the maximum norm.

Throughout we have been measuring distances in the usual Euclidean norm. However, it is possible that (2) is valid for a large class of norms.

Let l_∞ denote the distance induced by the maximum norm, i.e., given two points $a = (x, y), a' = (x', y') \in \mathbf{R}^2$, $l_\infty(a, a') = \max(|x - x'|, |y - y'|)$. Let Per_∞ denote the perimeter of a convex polygon measured in the l_∞-distance. We have the following

Theorem 2 *Let $\{c_0, \ldots, c_n\}, \{c'_0, \ldots, c'_n\}$ be two point sets in the plane satisfying $l_\infty(c_i, c_j) \le l_\infty(c'_i, c'_j)$ for all i and j. Then*

$$\mathrm{Per}_\infty \operatorname{conv}\{c_0, \ldots, c_n\} \le \mathrm{Per}_\infty \operatorname{conv}\{c'_0, \ldots, c'_n\}.$$

Proof: First we make an observation about the perimeter of a convex polygon in the maximum norm.

Lemma 3 *Given a finite point set P in the plane, let R denote the smallest rectangle enclosing P such that the angles between the sides of R and the coordinate axes are equal to $\pi/4$. Let a, b, c, d be (not necessarily distinct) elements of P such that each side of R contains at least one of them. Then*

$$\mathrm{Per}_\infty \operatorname{conv}(P) = \mathrm{Per}_\infty \operatorname{conv}\{a, b, c, d\} = \frac{\mathrm{Per}(R)}{\sqrt{2}}.$$

Proof: Omitted in this abstract. □

Corollary. *Let P be a finite point set in the plane, $P^* \subseteq P$. Then*

$$\mathrm{Per}_\infty \operatorname{conv} P^* \le \mathrm{Per}_\infty \operatorname{conv} P. \square$$

Next we show that it is sufficient to prove Theorem 2 for $n = 3$ (for 4 points). For $n < 3$ there is nothing to prove. Assume that the statement is true for $n = 3$, and let $n > 3$. Let a, b, c, d denote the points of $\{c_0, \ldots, c_n\}$ sitting on the boundary of the rectangle R enclosing P, as in Lemma 3. Let a', b', c', d' denote the corresponding points in $P' = \{c'_0, \ldots, c'_n\}$. Suppose, in order to obtain a contradiction, that

$$\mathrm{Per}_\infty \operatorname{conv} P > \mathrm{Per}_\infty \operatorname{conv} P'.$$

By Lemma 3 and by the above Corollary,

$$\mathrm{Per}_\infty \operatorname{conv}\{a, b, c, d\} = \mathrm{Per}_\infty \operatorname{conv} P >$$

$$\mathrm{Per}_\infty \operatorname{conv} P' \ge \mathrm{Per}_\infty \operatorname{conv}\{a', b', c', d'\},$$

contradicting our assumption that Theorem 2 is true for $n = 4$.

Suppose now that in the maximum norm distances between the elements of $P = \{a, b, c, d\}$ are at most as large as the corresponding distances within $P' = \{a', b', c', d'\}$. but

$$\mathrm{Per}_\infty \operatorname{conv} P > \mathrm{Per}_\infty \operatorname{conv} P'.$$

We can assume without loss of generality that a', b', c', d' are in convex position. Suppose not. Then one of them, say d', is in the interior of the triangle induced by the others. The l_∞-distance of d' from a', b' and c' is either their vertical distance or their horizontal distance. Suppose without loss of generality that at most one of these three distances is the horizontal distance. Then we can move d' horizontally until it hits the boundary of the triangle $a'b'c'$, so that none of the l_∞-distances decreases.

We can also assume that a, b, c, d are in convex position. Otherwise, if, say, d is in the triangle abc, then by the Corollary

$$\mathrm{Per}_\infty \operatorname{conv}\{a, b, c\} = \mathrm{Per}_\infty \operatorname{conv} P$$

$$> \mathrm{Per}_\infty \operatorname{conv} P' \ge \mathrm{Per}_\infty \operatorname{conv}\{a', b', c'\},$$

a contradiction.

Let $abcd$ be the cyclic order of the vertices of $\operatorname{conv} P$. By symmetry, there are only two essentially different cases.

- *Case A:* The cyclic order of the vertices of $\operatorname{conv} P'$ is $a'b'c'd'$. Then

$$\mathrm{Per}_\infty \operatorname{conv} P = l_\infty(a, b) + l_\infty(b, c) + l_\infty(c, d) + l_\infty(d, a)$$

$$\le l_\infty(a', b') + l_\infty(b', c') + l_\infty(c', d') + l_\infty(d', a')$$

$$= \mathrm{Per}_\infty \operatorname{conv} P'.$$

- *Case B:* The cyclic order of the vertices of conv P' is $a'c'b'd'$. By the triangle inequality for the maximum norm, $l_\infty(a, b) + l_\infty(c, d) \le l_\infty(a, c) + l_\infty(b, d)$. Hence,

$$\mathrm{Per}_\infty \operatorname{conv} P = l_\infty(a, b) + l_\infty(b, c) + l_\infty(c, d) + l_\infty(d, a)$$

$$\le l_\infty(b', c') + l_\infty(d', a') + l_\infty(a', c') + l_\infty(b', d')$$

$$= \mathrm{Per}_\infty \operatorname{conv} P'.$$

This completes the proof of Theorem 2.□

4 Related problems and generalizations.

Theorem 1 can readily be generalized to bounded infinite sets $S \subset \mathbf{R}^2$. A mapping $f : S \to \mathbf{R}^2$ is said to be a *contraction* if

$$|f(p) - f(q)| \leq |p - q| \text{ for all } p, q \in S.$$

Theorem 3 *Let f be a contraction of a bounded set $S \subseteq \mathbf{R}^2$. Then*

$$\text{Per conv} f(S) \leq \text{Per conv} S. \square$$

Evidently, similar results cannot be true for the *surface area* of the convex hull of higher dimensional sets. However, the above arguments immediately yield

Theorem 4 *Let f be a contraction of a bounded set $S \subseteq \mathbf{R}^n$. Then*

$$W_{n-1}(\text{conv} f(S)) \leq W_{n-1}(\text{conv} S),$$

where W_{n-1} is the $(n-1)$-st mean projection measure (the mean width) in n dimensions. $\square$

It might be interesting to notice that by slightly modifying the proof presented in Section 2, we can generalize Lemma 2 in two different directions.

Lemma 2' *Let $\{B_1, \ldots, B_k, B_{k+1}, \ldots, B_m\}$, $k < m$ be a collection of n-dimensional balls of arbitrary radii in $\mathbf{R}^n$, and let c_i denote the center of B_i. Let $\vec{v}$ be a vector with the property that translating $\{c_1, \ldots, c_k\}$ along $\vec{v}$, the distance between any c_i and c_j, $i \leq k < j$ does not decrease. Then*

$$(i) \quad \text{Vol}_n\left(\bigcup_{i=1}^{m} B_i\right) \text{ does not decrease,}$$

$$(ii) \quad \text{Vol}_n\left(\bigcap_{i=1}^{m} B_i\right) \text{ does not increase,}$$

during this translation. $\square$

Combining Lemma 1 and (the special case $k = 1$ of) Lemma 2'(ii), we obtain a proof of (6) for $m \leq n$, somewhat different from the one given in [G].

It is worth mentioning that conjecture (1) of Thue Poulsen, Kneser and Hadwiger cannot be generalized to other Minkowski planes. More precisely, we have the following.

Theorem 5 *Let the plane be equipped with a norm $||.||$ such that the unit disk $C = \{x \in \mathbf{R}^2 : ||x|| \leq 1\}$ is not an ellipse.*

Then one can find points $a, b, a', b' \in \mathbf{R}^2$ with the property that $||a - b|| < ||a' - b'||$ but

$$\text{Area}((C + a) \cup (C + b)) > \text{Area}((C + a') \cup (C + b')). \square$$

On the other hand, the weaker statement (2) might remain true for a large class of other norms substantailly different from the Euclidean one.

In the last section we have shown that (2) holds in the plane equipped with the maximum norm (when the unit disk is a square). A related question is the following.

Given two collections of axis parallel unit squares in the plane, $S_1, \ldots, S_n$ and $S'_1, \ldots, S'_n$, such that $\text{Area}(S_i \cup S_j) > \text{Area}(S'_i \cup S'_j)$ for all i and j. Is it true that $\text{Area}(\cup S_j) > \text{Area}(\cup S'_j)$?

Some related questions with applications to particle physics are discussed by Lieb and Simon in [L] and [LS].

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