Probing Polygons Minimally is Hard

(Extended abstract)

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Abstract

Let Γ be a set of convex unimodal polygons in fixed position and orientation. We prove that the problem of determining whether K probes are sufficient to identify all polygons P in Γ is NP-Hard for three different ways of orienting the probes. This implies that the same results hold for most interesting classes of polygons on which line probes can be used.

1 Introduction

A *probe* is a directed line l aimed at a polygon P. The result of a probe is a contact point p_l, which is on the boundary of P if l intersects P, and at infinity otherwise. Let Γ be a set of m convex n-gons in fixed position and orientation which all share a common vertex p. In [Lyo88], it is shown that for every such Γ, there exists a set Π of probes aimed at p, with $|\Pi| \leq m-1$, such that all polygons $P \in \Gamma$ can be identified by looking at the results of the probes in Π. It is also proved that sometimes one cannot do this with less than $m-1$ directions.

We recall that a polygon P is unimodal if the distance function from a vertex of P to all other vertices in clockwise order around the boundary of P has exactly one local maximum. In this abstract we show that the problem of deciding whether K probes are enough to identify $P \in \Gamma$ is NP-Hard for three different ways of orienting the probes (including the one used in [Lyo88]) even when the n-gons in Γ are not only convex, but also unimodal. This proof uses a reduction from the *Minimum Test Set* problem (see [GJ79]).

2 Generating convex unimodal n-gons

In this section we show how to generate up to 2^{n-2} strictly convex unimodal n-gons, while using only $O(n \log n)$ bits to represent the vertices of each polygon. This construction is used in the proofs of the following sections.

Let n be given, O denote the origin, and consider the part of the unit circle which lies in the first quadrant. Given a point p, we denote by $\theta(p)$ the angle between the positive direction of the x-axis and the line segment from O to p. For $k = 0, 1, \ldots, n-1$, define $\alpha_k = k\pi/(2n-2)$. Let us denote by ν_k the point on the unit circle at angle α_k, i.e. $\nu_k = (\cos\alpha_k, \sin\alpha_k)$, for $k = 0, 1, \ldots, n-1$ (note that even though ν_k and α_k are dependent on n, this dependence will be left implicit throughout this abstract to simplify the notation). For $k = 1, 2, \ldots, n-2$, let λ_k be the midpoint of the line segment from ν_{k-1} to ν_{k+1}, and let $r_k = \|\lambda_k\|$. We note that λ_k lies on the line segment from the origin to ν_k.

Lemma 2.1 *The value of r_k is independent of k and, provided $n \geq 3$,*

$$r_k < 1 - \frac{1}{(n-1)^2}$$

Proof : This can be shown by using the Taylor series expansion for $r_k = \cos(\pi/(2n-2))$.—qed

We are now ready to define polygons $P_0, \ldots, P_{2^{n-2}-1}$. Let us number the vertices of P_i from 0 to $n-1$, and denote the k^{th} vertex of P_i in counterclockwise order by $v_{i,k}$. We describe a point (x, y) by a pair (ρ, ϕ) where

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} && \text{norm of the vector } (x,y). \\ \phi &= \frac{2\theta((x,y))}{\pi} && \text{angle made by } (x,y) \text{ normalized.} \end{aligned}$$

We will number the bits of i from 0 to $n-3$ starting with the least significant bit; the k^{th} vertex of P_i is set to ν_k if the $(k-1)^{\text{st}}$ bit of i is 0, and to some point very close to λ_k otherwise. More formally, the coordinate vector of the k^{th} vertex of P_i in our new coordinate system is

$$v_{i,k} = \begin{cases} (1,0), & \text{if } k = 0; \\ (1,1), & \text{if } k = n-1; \\ \left(1, \dfrac{2\alpha_k}{\pi}\right), & \text{if } i \text{ div } 2^{k-1} \equiv 0 \pmod 2; \\ \left(1 - \dfrac{1}{(n-1)^2}, \dfrac{2\alpha_k}{\pi}\right), & \text{if } i \text{ div } 2^{k-1} \equiv 1 \pmod 2. \end{cases}$$

Lemma 2.2 *P_i is strictly convex for $i = 0, 1, \ldots, 2^{n-2}-1$.*

Proof : The proof of this fact will be omitted here. □

Lemma 2.3 *P_i is unimodal for $i = 0, 1, \ldots, 2^{n-2}-1$.*

Proof : Suppose that $j < k$, and consider the triangle $(v_{i,j}, v_{i,k}, v_{i,k+1})$. By construction, the angle that the line through two consecutive vertices of P_i makes with the positive direction of the x-axis is in the range $(\pi/2, \pi)$. Furthermore, since P_i is convex, the angle made by the line through $v_{i,j}$ and $v_{i,k}$ is strictly smaller than the angle made by the line through $v_{i,k}$ and $v_{i,k+1}$. Thus $\theta > \pi/2$, and so $\cos\theta < 0$, and $L^2 = l^2 + d^2 - 2ld\cos\theta > l^2 + d^2 > l^2$. Hence $d(v_{i,j}, v_{i,k+1}) > d(v_{i,j}, v_{i,k})$. This is true for all i, j, k provided $j < k$, and hence the distances from $v_{i,j}$ to $v_{i,j}, v_{i,j+1}, \ldots, v_{i,n-1}$ are strictly increasing. By symmetry, the distances from $v_{i,j}$ to $v_{i,j}, v_{i,j-1}, \ldots, v_{i,0}$ are also strictly increasing. Since this holds for all i, j, we conclude that P_i is unimodal. □

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Lemma 2.4 *P_i can be represented using at most $O(n \log n)$ bits for $i = 0, 1, \ldots, 2^{n-2} - 1$.*

Proof : We prove that each coordinate of each vertex of P_i can be expressed as a rational number (using our coordinate system) in a way such that both the numerator and the denominator have value at most $O(n^2)$. This is clear in the case of the first coordinate. Consider now the second coordinate. It has value $2\theta(v_{i,k})/\pi = k/(n-1)$. Since both the numerator and the denominator are less than n, it can be expressed using $O(\log n)$ bits as well. This implies that each $v_{i,k}$ can be represented using at most $O(\log n)$ bits, and so we conclude that P_i can be expressed using $O(n \log n)$ bits as required. □

Combining Lemmas 2.2, 2.3 and 2.4, we get :

Theorem 2.1 *For $i = 0, 1, \ldots, 2^{n-2} - 1$, P_i is a strictly convex unimodal n-gon which can be represented using at most $O(n \log n)$ bits.*

For the remainder of this abstract, we will express points in ordinary polar coordinates (ρ, θ) unless otherwise specified, as this is more intuitive than the transformed coordinates used to represent the polygons.

3 Probing convex polygons

We now want to show how the polygons in which we are interested are probed. Consider a line L in the plane, and a point $p \in L$. Let Γ be a set of convex n-gons $P_1, \ldots, P_m$. Assume that each polygon P is positioned such that one of its edges, denoted by $e[P]$, is collinear with L, and that the left endpoint of $e[P]$, denoted by $l[P]$, coincides with p.

We want to determine a set of rays $\Pi = \{\sigma_1, \ldots, \sigma_k\}$ such that we can differentiate between two polygons P_i, P_j in Γ using the contact points obtained by probing them with rays in Π. In [Lyo88], it is shown that

Theorem 3.1 (Lyons) *$m - 1$ probes are always sufficient and sometimes necessary to identify an object $P \in \Gamma$ positioned such that $l[P]$ is coincident at p and $e[P]$ is collinear with L.*

In [Lyo88], an algorithm which finds a set of at most $m - 1$ such rays in $O(mn + m^2)$ time is given. All rays in Π are aimed at p through another vertex of some polygon $P \in \Gamma$. This algorithm does not guarantee that the set of directions found has minimal cardinality. We will show that the problem of determining whether K rays are enough to identify an object $P \in \Gamma$ is NP-Hard for three different ways of choosing our probes, namely the following :

C1. All probes $\sigma_1, \ldots, \sigma_K$ are aimed at p through at least one other vertex of some polygon $P \in \Gamma$ – all probes generated by the algorithms in [Lyo88] are of this kind.

C2. All probes $\sigma_1, \ldots, \sigma_K$ are aimed at p, without necessarily going through a vertex of a polygon $P \in \Gamma$.

C3. The probes used are completely arbitrary.

The formal definition in the style of [GJ79] for the three corresponding problems which we will prove NP-Hard is the following :

Instance : A set Γ of convex unimodal polygons in fixed position and orientation, with a common vertex p and the edge clockwise to p collinear with a fixed line L. An integer K such that $1 \leq K \leq |\Gamma| - 1$.

Question : Is there a set or probes $\Pi = \sigma_1, \ldots, \sigma_k$ ($k \leq K$) satisfying condition C1, C2 or C3 above (depending on the version of the problem which is considered) which allows us to identify all polygons $P \in \Gamma$?

The first two versions of the problem are in fact NP-complete. It is not known whether the third one belongs to NP or not.

4 Probing towards a fixed point p belongs to NP

In this section, we show that the first two of the three problems enumerated above belong to NP. We assume that each polygon is given by a list of the cartesian coordinates of its vertices in clockwise order.

Lemma 4.1 *The version of the probing problem in which the probes have to satisfy condition C1 belongs to NP.*

Proof : The idea is to guess the coordinates of the vertices towards which the probes are aimed. □

To show that the version of the probing problem in which the probes only have to satisfy condition C2 is in NP, we need two definitions.

Definition 4.1 *A* special point *is either a vertex of P for some $P \in \Gamma$, or the intersection of an edge of P_i with an edge of P_j which is not collinear with it, for $P_i, P_j \in \Gamma$.*

Definition 4.2 *A* target *is either is special point, or the midpoint of the line segment joining two special points adjacent in the an ordering of all special points by polar angle around p.*

We note that there are at most $O(m^2 n^2)$ such targets for all sets Γ of m convex n-gons, and that each of them can be represented as a rational number using a number of bits polynomial in the size of the input.

Lemma 4.2 *The version of the probing problem in which the probes have to satisfy condition C2 belongs to NP.*

Proof : Suppose that there is a set Π of K probes which are directed towards p and identify all $P \in \Gamma$. Look at all targets, sorted by polar angle around p. It can be checked that each probe aimed towards p through an edge of some $P_i \in \Gamma$ is equivalent to a probe aimed towards p through a target, i.e. that these two probes meet the same edges of the same polygons in the same order (although at slightly different places). Hence we can guess the second point determining a ray in Π as being a target, and check this guess in polynomial time. □

Note that the crucial fact in this proof is the possibility of ordering a small number of targets, which can all be represented using only a polynomial number of bits, by polar angle around p. If there is no such point p towards which all the probes are directed, as in the case where the probes only have to satisfy condition C3, the problem becomes much harder and determining whether it belongs to NP or not remains an open problem.

5 Reduction from Minimum Test Set

We now want to give a way to reduce the *Minimum Test Set* problem (see [GJ79]) to our probing problems (the reduction will be the same in all three cases). First recall the statement of the *Minimum Test Set* problem :

Instance : A collection $C = \{C_1, \ldots, C_l\}$ of subsets of a finite set S, and a positive integer $K \leq |C|$.

Question : Is there a subcollection $C' \subseteq C$ with $|C'| \leq K$ such that for each pair of distinct elements $u, v \in S$ there is some set $c \in C'$ that contains exactly one of u and v?

Without loss of generality, we assume that the elements of S are numbered from 1 to $|S|$, that $|C| > 1$ and that $\emptyset \notin C$. Let $n = 2|C| + 1$. We choose p to be the point $(1,0)$, and L to be the line through p with slope -1. For each $i \in S$, let $p(i) = \sum_{\{C_j | i \in C_j\}} 2^{2j-2}$, and let $P_{p(i)}$ correspond to i, where $P_{p(i)}$ is taken from a list $P_0, \ldots, P_{2^{n-2}-1}$ of strictly convex n-gons generated as in Section 2.

Intuitively, this means that we have one polygon for each element in S, and that every other vertex corresponds to a subset of S which is in C. For all i and k, $v_{p(i),2k} = \nu_{2k}$, and $v_{p(i),2k-1}$ is equal to ν_{2k-1} if element i of S is in C_k, but is a little bit closer to the origin if i is not in C_k.

Lemma 5.1 *The transformation from* Minimum Test Set *can be done in polynomial time.*

For notational convenience, let us assume that $p(i) = i$ for $i = 1, 2, \ldots, |S|$, so that $P_{p(i)}$ can be expressed more simply as P_i. We conclude this section by making two observations.

Lemma 5.2 *Let P_i, P_j be two polygons in $P_0, \ldots, P_{2^{n-2}-1}$. If a probe σ aimed at p hits P_i between $v_{i,2k-2}$ and $v_{i,2k}$, it contacts P_j between $v_{j,2k-2}$ and $v_{j,2k}$. Furthermore, if σ is aimed at p through $v_{i,2k-1}$, it contacts P_j at $v_{j,2k-1}$ if and only if $v_{j,2k-1} = v_{i,2k-1}$.*

Proof: By construction, $v_{i,2k-2} = v_{j,2k-2}$ and $v_{i,2k} = v_{j,2k}$, and by Theorem 2.1, P_i and P_j are convex. Thus, since σ hits P_j, it must do so between $v_{j,2k-2}$ and $v_{j,2k}$. Suppose now that σ hits P_i at $v_{i,2k-1}$. If $v_{i,2k-1} = v_{j,2k-1}$, then σ also contacts P_j at $v_{j,2k-1}$. On the other hand, if the contact point of σ with P_i is different from $v_{i,2k-1}$, then since $v_{i,2k-1}$, $v_{j,2k-1}$ and p are not all collinear (since $v_{i,2k-1}$ and $v_{j,2k-1}$ are collinear with the origin, but do not lie on the x-axis), σ does not contact P_j at $v_{j,2k-1}$ either, as required. □

Lemma 5.3 *Let P_i, P_j be two polygons in $P_0, \ldots, P_{2^{n-2}-1}$. If a probe σ satisfying condition C1 distinguishes between P_i and P_j, then $\exists k$ such that σ hits either $v_{i,2k-1}$ or $v_{j,2k-1}$.*

Proof : Since σ satisfies condition C1, Lemma 5.2 implies that $\exists P_l$ which is hit by σ at $v_{l,2k-1}$. As $v_{i,2k-1} \neq v_{l,2k-1}$, and there are only two possible values for the k^{th} vertex of polygons in Γ, this means that either $v_{l,2k-1} = v_{i,2k-1}$, or $v_{l,2k-1} = v_{j,2k-1}$, i.e. that σ hits one of $v_{i,2k-1}$ or $v_{j,2k-1}$, as required. □

6 Proof of correctness

To prove that our problems are NP-Hard, we now need to show that there is a set Π of probes satisfying our conditions if and only if there is a subset $C' \subseteq C$ with $|C'| \leq K$ such that, for each pair $u, v \in S$, $\exists C(u,v) \in C'$ to which exactly one of u and v belongs. First we prove :

Lemma 6.1 *Each solution to* Minimum Test Set *gives a solution to the corresponding probing problem under each one of our three conditions.*

Proof : We show that, under our hypothesis, there exists a set of probes satisfying condition C1 which can identify all $P \in \Gamma$. Suppose that $\forall u, v \in S$, $\exists C(u,v) \in C'$ such that exactly one of u and v is in $C(u,v)$. For each k, define σ_k to be the ray starting at infinity and aimed towards p through ν_{2k-1}. We claim that $\Pi = \{\sigma_k \mid C_k \in C'\}$ is a set of probes satisfying condition C1 which identifies all $P \in \Gamma$. Since $C_k \neq \emptyset$, ν_{k-1} is a vertex of at least one $P \in \Gamma$, and therefore all probes in Π satisfy condition C1. Given $P_i, P_j \in \Gamma$, consider $i, j \in S$. By hypothesis, there is $C(i,j) \in C'$ (say C_m) such that exactly one of i and j is in $C(i,j)$. Thus, by construction, $v_{i,2m-1} \neq v_{j,2m-1}$. Therefore, by Lemma 5.2, probe $\sigma_m \in \Pi$ distinguishes between P_i and P_j. This holds for all pairs P_i, P_j of elements of Γ, and so Π allows us to identify all $P \in \Gamma$. Clearly $|\Pi| = |C| \leq K$, and hence this proves the lemma. □

We now want to show that, if there is a set of probes satisfying one of our conditions, then it gives a solution to the instance of *Minimum Test Set*. We first prove it when condition C1 is satisfied by the probes, and we then generalize this to the other kinds of probes, namely those satisfying only condition C2 or C3.

Lemma 6.2 *If there exists a set $\Pi = \sigma_1, \ldots, \sigma_l$ of probes satisfying condition C1, with $l \leq K$, and which can identify all $P \in \Gamma$, then there is a subset $C' \subseteq C$ with $|C'| \leq K$, such that $\forall u, v \in S$, $\exists C(u,v) \in C'$ to which exactly one of u and v belongs.*

Proof : By condition C1, each probe $\sigma \in \Pi$ is aimed at p from infinity through some vertex $v_{i,k}$. Let $C' = \{C_k \mid \sigma \in \Pi$ goes through $v_{i,2k-1}$ for some $0 \leq i < |S|\}$. Clearly $|C'| = |\Pi| \leq K$. Consider $i, j \in S$. By the definition of Π, condition C1 and Lemma 5.3, $\exists \sigma \in \Pi$ which distinguishes between polygons P_i and P_j and is aimed at $v_{i,2k-1}$ or $v_{j,2k-1}$ for some k (say $v_{i,2k-1}$). By Lemma 5.2, $v_{i,2k-1} \neq v_{j,2k-1}$, and so by construction exactly one of i and j is in C_k. This holds for all pairs of elements of S, and so C' is the required solution to the *Minimum Test Set* problem. □

Lemma 6.3 *If there exists a set $\Pi = \sigma_1, \ldots, \sigma_k$ $(k \leq K)$ of probes satisfying condition C2 or condition C3 which can identify all $P \in \Gamma$, then there is a subset $C' \subseteq C$ with $|C'| \leq K$, such that $\forall u, v \in S$, $\exists C(u,v) \in C'$ for which exactly one of u and v is in $C(u,v)$.*

Proof : We show that, for each set $\Pi = \sigma_1, \ldots, \sigma_k$ of probes satisfying condition C3 which can identify all $P \in \Gamma$, we can define a set $\Pi' = \sigma'_1, \ldots, \sigma'_k$ of probes which satisfy condition C1 and can identify all $P \in \Gamma$. Consider $\sigma \in \Pi$. We construct a probe σ' which will replace σ in Π'. Without loss of generality assume that σ does not miss all polygons in Γ, that it hits P_i, and that there is $P_{i'} \in \Gamma$ which it does not hit at the same place as P_i (or maybe that it misses altogether). Suppose that it hits P_i between $v_{i,2k}$ and $v_{i,2k+2}$. By Lemma 5.2, σ must hit $P_{i'}$ between $v_{i',2k}$ and $v_{i',2k+2}$. There are only 8 different ways in which this can happen (here we assume without loss of generality that $\|v_{i,2k-1}\| > \|v_{i',2k-1}\|$) :

1. σ hits an edge of P_i and the edge of $P_{i'}$ on the same side of the line segment from $v_{i,2k-1}$ to $v_{i',2k-1}$.
2. σ hits an edge of P_i and $v_{i',2k-1}$.
3. σ hits an edge of P_i and the edge of $P_{i'}$ on the other side of the line segment from $v_{i,2k-1}$ to $v_{i',2k-1}$.
4. σ hits an edge of P_i and the vertex of $P_{i'}$ on the other side of the line segment from $v_{i,2k-1}$ to $v_{i',2k-1}$ (not $v_{i',2k-1}$).
5. σ hits an edge of P_i and misses $P_{i'}$.
6. σ hits $v_{i,2k-1}$ and hits $P_{i'}$ on either $v_{i',2k-2}$ or $v_{i',2k}$.
7. σ hits $v_{i,2k-1}$ and hits $P_{i'}$ inside one of the two edges adjacent to $v_{i',2k-1}$.
8. σ hits $v_{i,2k-1}$ and $v_{i',2k-1}$.

In each case, we let σ' be the probe aimed at p through $v_{i,2k-1}$. If σ distinguished between P_i and P_j, obviously so does σ'. This holds for all $i, j \leq |S|$, and thus Π' identifies all $P \in \Gamma$. By Lemma 6.2, this implies that the required subset C' of C exists. □

Combining Lemmas 4.1, 4.2, 5.1, 6.1, 6.2 and 6.3 on one side, and Lemmas 5.1, 6.1 and 6.3 on the other, we get :

Theorem 6.1 *The versions of the probing problem in which the probes have to satisfy conditions C1 or C2 are NP-Complete. The version of the problem in which arbitrary probes are allowed is NP-Hard.*

We recall that a polygon P is weakly externally visible if, for all q on the boundary of P, there is a ray σ which has endpoint q and intersects P only at q. Clearly the class of weakly externally visible polygons contains all classes of polygons which can be identified using line probes, since we may not be able to probe part of the boundary of P if it is not weakly externally visible. We note that, in this more general case, the point p will be chosen on the convex hull of P, and $e[P]$ will be an edge of the convex hull of P adjacent to p.

Let $\mathcal{P}$ be a class of polygons which contains all convex unimodal polygons and which is contained in the class of weakly externally visible polygons. Since the polygons generated by the reduction were both strictly convex and unimodal, Theorem 6.1 can be applied to class $\mathcal{P}$ and we get :

Theorem 6.2 *Let Γ be a set of polygons belonging to a class $\mathcal{P}$ which satisfies the two conditions above. The versions of the probing problem in which the probes have to satisfy conditions C1 or C2 are NP-Complete, and the version in which arbitrary probes are allowed is NP-Hard.*

7 Conclusion

We have shown that the problem of selecting probes to uniquely identify a polygon belonging to a set Γ of convex unimodal n-gons in fixed position an orientation is NP-Complete under two criteria for the choice of probes, and that it is NP-Hard under a third, more general criterion – namely that no probe is invalid. We do not know whether this third case belongs to NP or not. It is also interesting to note that we can extend the convex unimodal polygons generated from the *Minimum Test Set* problem into cylinders into higher dimensions without affecting the reduction or its proof of correctness. Thus, the NP-Hardness results also generalize to higher dimensions. Finally, they furthermore imply that these problems are also NP-Hard for all other classes of polygons of which either convex or unimodal polygons form a subclass, and which are subclasses of weakly externally visible polygons.

References

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