IMMOBILIZING FIGURES ON THE PLANE

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Let S be a bounded connected open set on the plane (henceforth called plane set). A collection of points P on the boundary $\beta(S)$ are said to immobilize S if any "small" rigid movement of S causes one point in P to penetrate into S. It is known that any plane set S that is not an open disk, can be immobilized with at most four points [1]. When S is the interior of a parallelepiped, four points are needed. Immobilization problems were introduced by W. Kuperberg [2] and also appeared in [5]. Applications of immobilization problems can be found in robotics, specially in grasping problems, see [3,4]. In this paper we prove in the affirmative a conjecture of Kuperberg, namely we prove:

Theorem 1: Any plane set with smooth boundary can be immobilized with three points.

Some definitions and terminology will be needed before we can give a sketch of our proof. Consider three points $\{x_1, x_2, x_3\}$ on the boundary $\beta(S)$ of S. We now proceed to give conditions under which $\{x_1, x_2, x_3\}$ immobilize S.

For x_1, x_2, x_3 to immobilize S, the following two conditions must be satisfied:

1) The normals to $\beta(S)$ at x_1, x_2, x_3 must all meet at a point **P**; see [1]. We may assume w.l.o.g that **P** is the origin **0**.
2) The three tangents to S at x_1, x_2, x_3 must define a (bounded) triangle with vertices y_1, y_2 and y_3 as shown in Figure 1.

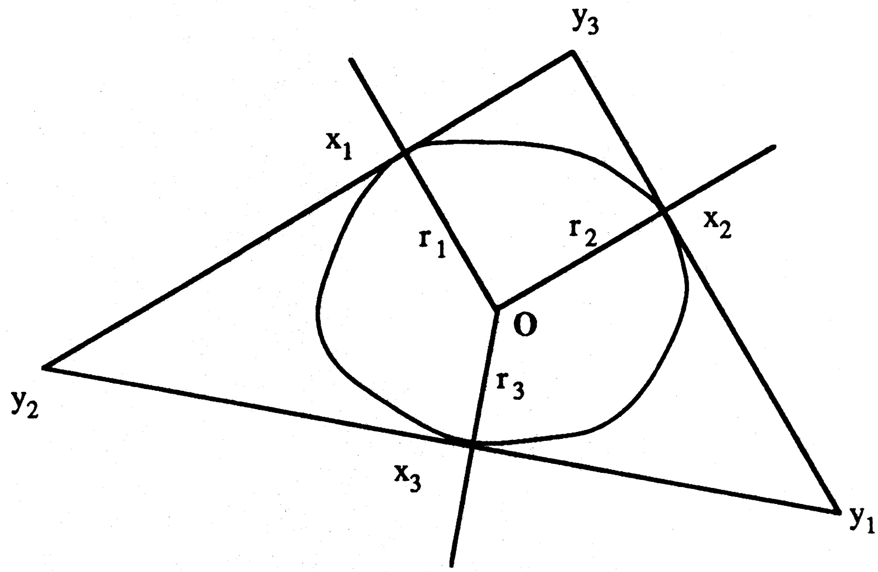

Figure 1

It is easy to see that Conditions 1 and 2 are not sufficient to guarantee that x_1, x_2 and x_3 immobilize S. We need to give a third condition that needs to be satisfied so that we can guarantee that they immobilize S. Some additional definitions will be needed.

Let κ_i be the *curvature* of $\beta(S)$ at x_i, r_i be the norm of x_i, and a_i the baricentric coordinates of the origin with respect to the triangle with vertices y_1, y_2 and y_3, i=1,2,3.

Theorem 2. Let x_1, x_2 and x_3 be three points in $\beta(S)$ such that 1) and 2) are satisfied. Let $s = a_1 r_1 \kappa_1 + a_2 r_2 \kappa_2 + a_3 r_3 \kappa_3$. Then if $s<1$, x_1, x_2 and x_3 immobilize S. If $s>1$, then x_1, x_2 and x_3 do not immobilize S.

Sketch of Proof. Using elementary tools in differential calculus, it is easy to prove that there is a differentiable reparametrization $\alpha(\theta)$ of $\beta(S)$ and a real differentiable function $\varphi(\theta)$ defined in a small neighborhood of 0 such that:

i) $\alpha(0) = x_1$ and $\alpha(\varphi(0)) = x_2$

ii) $|\alpha(\theta) - \alpha(\varphi(\theta))| = | x_1 - x_2|$

iii) The angle formed between the line through x_1 and x_2 and that by $\alpha(\theta)$ and $\alpha(\varphi(\theta))$ is precisely θ.

Let L_θ be the rigid mapping (the isometry) that maps the line segment $[x_1, x_2]$ to the segment $[\alpha(\theta), \alpha(\varphi(\theta))]$ and L^{-1}_θ be its inverse rigid mapping. $L^{-1}_\theta(\beta(S))$ can be visualized as the final position of S when we slide it over x_1 and x_2 so that the points $\alpha(\theta), \alpha(\varphi(\theta))$ are mapped into x_1 and x_2 respectively (θ in a small neighborhood $(-\delta, \delta)$ around 0).

Thus if $x_3 \in L^{-1}_\theta(S)$, for all $\theta \in (-\delta, \delta)$, δ sufficiently small, then x_1, x_2, x_3 **do** immobilize S. Similarly if $x_3 \notin L^{-1}_\theta(S)$, for all $\theta \in (-\delta, \delta)$, δ sufficiently small, then x_1, x_2, x_3 **do not** immobilize S.

It is easy to show that any rigid mapping can be decomposed into a rotation of the plane by an angle θ around the origin followed by a translation. Thus for every **x** on the plane, we may rewrite L_θ as follows: $L_\theta(\mathbf{x}) = r_\theta(\mathbf{x}) + X(\theta)$.

Thus $\alpha(\theta) = L_\theta(x_1) = r_\theta(x_1) + X(\theta)$ and $\alpha(\varphi(\theta)) = L_\theta(x_2) = r_\theta(x_2) + X(\theta)$.

Taking derivatives, we get:

$$\frac{d\,\alpha(\theta)}{d\theta} = \frac{d\,r_\theta(x_1)}{d\theta} + \frac{d\,X(\theta)}{d\theta} \qquad (1)$$

$$\frac{d\,\varphi(\theta)}{d\theta} = \frac{d\,r_\theta(x_2)}{d\theta} + \frac{d\,X(\theta)}{d\theta} \qquad (2)$$

We now show that $\frac{d\,X(\theta)}{d\theta} = 0$.

Evaluating, we get

$$\left.\frac{d\,\alpha(\theta)}{d\theta}\right|_0 = \left.\frac{d\,r_\theta(x_1)}{d\theta}\right|_0 + \left.\frac{d\,X(\theta)}{d\theta}\right|_0 \quad \text{and} \quad \left.\frac{d\,\varphi(\theta)}{d\theta}\right|_0 = \left.\frac{d\,r_\theta(x_2)}{d\theta}\right|_0 + \left.\frac{d\,X(\theta)}{d\theta}\right|_0$$

But since by definition x_i is orthogonal to the tangent to $\beta(S)$ at x_i, we have:

$\left.\frac{d\,\alpha(\theta)}{d\theta}\right|_0$ and also to $\left.\frac{d\,r_\theta(x_2)}{d\theta}\right|_0$ are orthogonal to x_i.

We have by calculating the inner product of (1) and (2) with x_1 and x_2 that

$$\left.\frac{d\,X(\theta)}{d\theta}\right|_0 = 0.$$

In general the curvature $\kappa(x)$ of $L_\theta(x)$ at a point x is given by: $-\kappa(x)\,|x|^3 = < x(\theta)\,, x > + \,|x|^2$. Then:

$$-\kappa(x_1)\, r_1^3 = < x^{\cdot\cdot}(0)\,, x_1 > + \, r_1^2,$$

$$-\kappa(x_2)\, r_2^3 = < x^{\cdot\cdot}(0)\,, x_2 > + \, r_2^2, \qquad \ldots\ldots \quad (3)$$

$$-\kappa(x_3)\, r_3^3 = < x^{\cdot\cdot}(0)\,, x_3 > + \, r_3^2.$$

Note that since $L_\theta(x_1) = \alpha(\theta)$ and $L_\theta(x_2) = \alpha(\varphi(\theta))$ then $\kappa(x_i) = \kappa_i$, $i = 1,2$.

Let b_i, $i=1,2,3$ be such that $b_1 + b_2 + b_3 = 1$ and $b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$, i.e. the baricentric coordinates of the origin with respect to the triangle determined by x_1, x_2 and x_3.

Using the coefficients b_i and (3) we get (multiplying by the corresponding b_i and adding):

$$-(b_1 r_1^2 \kappa(x_1) r_1 + b_2 r_2^2 \kappa(x_2) r_2 + b_3 r_3^2 \kappa(x_3) r_3) = <X(0)\,, \mathbf{0}> - (b_1 r_1^2 + b_2 r_2^2 + b_3 r_3^2).$$

Let $$a_i = \frac{b_i r_i^2;}{b_1 r_1^2 + b_2 r_2^2 + b_3 r_3^2}, \quad i=1,2,3.$$

It can be proved that the a_i's are precisely the baricentric coordinates of the origin with respect to the triangle with vertices y_1, y_2 and y_3.

Then $a_1 r_1 \kappa(x_1) + a_2 r_2 \kappa(x_2) + a_3 r_3 \kappa(x_3) = 1$. Suppose now that $a_1 r_1 \kappa(x_1) + a_2 r_2 \kappa(x_2) + a_3 r_3 \kappa(x_3) = 1 < a_1 r_1 \kappa_1 + a_2 r_2 \kappa_2 + a_3 r_3 \kappa_3$, then $\kappa(x_3) < \kappa_3$ and we can slide S over x_1 and x_2 leaving x_3 outside of S. Then x_1, x_2 and x_3 do not immobilize S. Conversely if $s < 1$, then $\kappa(x_3) > \kappa$, and $x_3 \in L^{-1}_\theta(S)$. In this case they do immobilize S. This concludes the proof of Theorem 2.

We proceed now to prove our main theorem.

Proof of Theorem 1. Consider the largest (open disk) D contained in S. Let C be the circle defined by the boundary of D. If C intersects $\beta(S)$ in three points not contained in a half circle of C, then we can immobilize S with three points [1]. Suppose then that C meets $\beta(S)$ at exactly two points x_a and x_b. Clearly the line segment joining x_a to x_b is a diameter of C. Suppose w.l.o.g that the radius of C is 1.

If $\kappa_a + \kappa_b < 2$, then it is easy to prove using Theorem 2 that there is a point z in $\beta(S)$ and two points x and y sufficiently close to x_a and x_b respectively which immobilize S. The critical case is when $\kappa_a = \kappa_b = 1$. It is then harder, but still possible to prove, again using Theorem 2, that x_a and two points x and y sufficiently close to x_b, as shown in Figure 2, immobilize S.

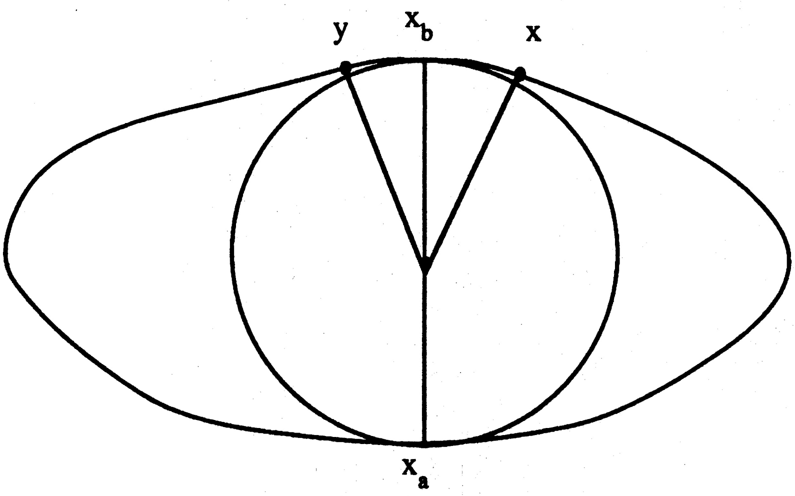

Figure 2

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