IMMOBILIZING A SHAPE IN THE PLANE
(Extended abstract)

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Abstract

Let P be any simply-connected shape in the plane that is not a circular disk. We say that a set of points I on the boundary of P immobilize the shape if any rigid motion of P in the plane causes at least one point of I to penetrate the interior of P. We prove that four points always suffice to immobilize any shape.

1. Introduction

The set of points I is said to immobilize a planar shape P if any rigid motion of P in the plane forces at least one point of I to penetrate the interior of P. Clearly, any minimal I contains only points belonging to the boundary of P. The disk is excluded from consideration since any number of points on its boundary leave it free to rotate.

Problems of immobilization of planar shapes were introduced by W. Kuperberg [K] and later reported in [O] where a number of open questions were presented:

- Do four points always suffice to immobilize any shape? Any convex shape?
- Find all the classes of convex shapes for which three points do not suffice.
- Do three points suffice for all smooth convex shapes?
- Design an algorithm for finding a set of immobilizing points for a given polygon.
- Extend to three (and higher) dimensions.

In this paper we answer the first of the above questions. The results are applicable in robotics, especially in the problem of grasping. [MNP] and [MSS] study a related problem of closure grasp for piecewise smooth objects, i.e. ability to respond to any external force or torque by applying appropriate forces at the grasp points. The ideas of using the inscribed circle and Voronoi diagram, exploited in our paper, were first used in [BFG], while the idea of normals to the boundary of a triangle meeting at a point appears in [MP], all in the context of an equilibrium grip. The problem of immobilization, that differs from all known variations of grasping, is studied here for any shape bounded by a Jordan curve.

A rigid motion of a set S on the plane is a mapping M from the set $t \times S$ (t represents time) to the plane, continuous with respect to its first coordinate, such that for every pair of points $u, v \in S$ the distance between their images remains constant for all t and $M(0, u)=u$ for every element of S.

By *shape* we mean a set bounded by a Jordan curve. If the shape P admits an inscribed circle (not necesserily the largest one) meeting the boundary of P in a set such that its convex hull contains the center of the circle, then some three points are sufficient to immobilize P. Otherwise four points may be needed.

For most proofs in this extended abstract only general lines are presented or they are completely omitted. Although some of the statements seem intuitively clear they often require involved and subtle proofs. The interested reader is invited to consult [CSU] for complete proofs.

2. Immobilizing a shape.

Given a shape P, let S be a locally largest inscribed circle of P, and let O be the center of S. There are two possible cases:

1) One of such circles S touches P in three points A_1, A_2, and A_3 such that O is an interior point of triangle $A_1A_2A_3$ (see Fig. 1). In this case we call P a 3-type shape.

2) If P is not a 3-type shape then any locally largest circle S is a diameter circle, i.e. touches P in the endpoints of a diameter of S (an example is the intersection of two disks; see also figures in [O]).

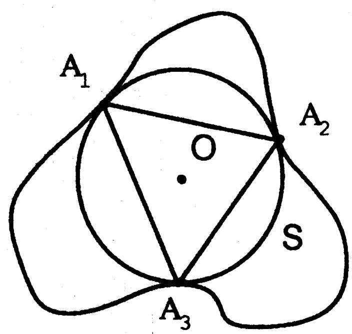

Figure 1

Theorem 1. If P is a 3-type shape then P has an immobilizing set consisting of three points.

Proof. It is possible to choose the points on the boundaries of S and P, such that O belongs to the interior of the triangle determined by them and, as immobilization points, they disallow any rotation around O. We prove that these points disallow any other movement as well. In the remaining case any possible motion of P will move O to a new position O'≠O in the neighborhood of O. Let S' be the new position of S, and let S and S' intersect at points U and V; U and V lie on the bisector b of the segment OO' (see Fig. 2) which separates the boundary of S into two halves, one being inside and one being outside S', respectively.

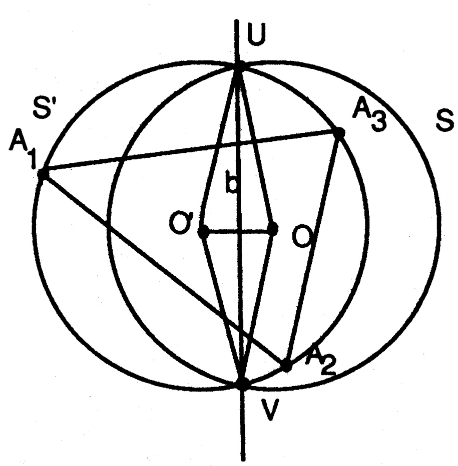

Figure 2

When O' is near O, OO' lies completely inside triangle $A_1A_2A_3$. In that case b intersects triangle $A_1A_2A_3$, therefore splitting points A_1, A_2 and A_3; there is at least one point on each side of b, thus at least one in the interior of S', and that one penetrates S. Therefore O cannot move. ♦

Consider now diameter shapes. For a given shape P let U and V be touching points of a diameter circle S, centered at O (suppose, for simplicity, that UV is vertical, both with x-coordinate equal to 0, and V being below U). In order to immobilize P, we may restrict the analysis to some ∂-neighborhood of U and V only; clearly any set that will immobilize restricted shape will also immobilize P. Thus ∂-interval of P consists of two continuous pieces, upper P_u and lower P_l (containing U and V, respectively) such that each point on them has x-coordinate between $-\partial$ and ∂; each P_u and P_l is further subdivided into its left and right portion by point U or V.

Let f(A) be the radius of the largest circle centered at A, which touches both P_u and P_l. As A must be equidistant from P_u and P_l, f is defined only for some points between P_u and P_l. There are two cases:

a) for every $\varepsilon>0$ there exists a point A, such that $|AO|<\varepsilon$, f is defined for A and $f(A)<f(O)=|OU|$. In other words, S cannot move from its original position O, without intersecting the exterior of P; we refer to such a shape as a non-tube,

b) there exists $\varepsilon>0$ such that $f(A)=f(O)$ for any center A for which f is defined and such that $|AO|<\varepsilon$ (intuitively in this case S may slide inside P in some ε-neighborhood of O); we refer to such a shape as a tube.

Theorem 2. Four points always suffice to immobilize any diameter non-tube shape.

Proof. To immobilize the shape P, we choose a set I of four points U', V', U", and V" on P, one on each left and right portion of P_u and P_l, such that U' and V' (U" and V") are touching points of an inscribed circle S' (S") centered at O' (O", respectively) with P_u and P_l, where O' and O" are in the "neighborhood" of O and lying on opposite sides of the line UV (as shown in Fig. 3).

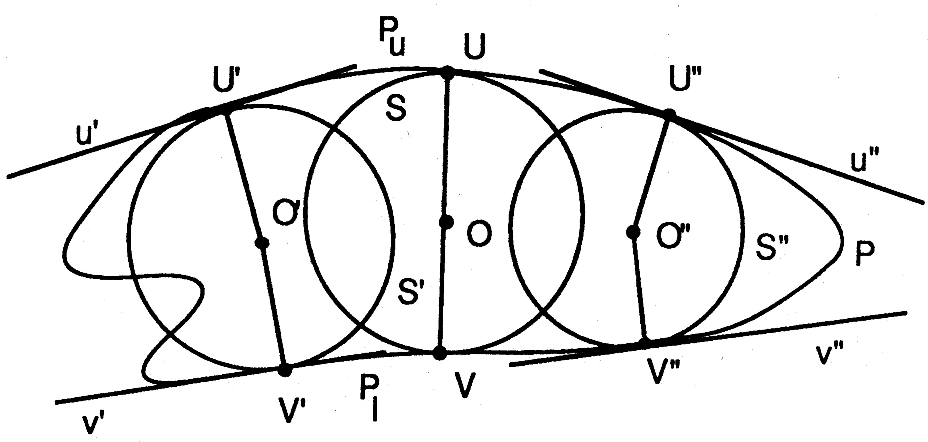

Figure 3

By u', v', u", and v" we denote the four tangents to S' and S" (they are not necessarily tangent to P). We show in [CSU] that for any $\varepsilon>0$ we can choose $\partial>0$ such that the following properties are satisfied on ∂-intervals:

(i) the slopes of the tangents u', v', u", and v" (the angles the tangents form with the x-axis) are between $-\varepsilon$

and ε (the choice of any $\varepsilon<\pi/4$ will suffice for our purpose),

(ii) the slope of O'O" is between -ε and ε (the choice of any $\varepsilon<\pi/4$ will also suffice),

(iii) two tangents u' and v' at the touching points U' and V' of S' with P_u and P_l intersect to the left of UV (or, equivalently, the angle U'O'V' in the polygon UU'O'V'V is >π; we refer to this angle as *critical* angle at O'); analogously two tangents u" and v" intersect to the right of UV.

We may assume that P and S do not share many points in a neighborhood of U or V, otherwise either P would be a 3-type shape or we could replace one of S' or S" by S. The proof that the four points immobilize P uses the fact that any movement of P preserves the distance between O' and O". Let m' and m" be two arcs starting at O' such that the tangents at O' to these arcs are parallel to u' and v', respectively. The reader may check that for possible movement of O' we get the region limited by m' and m" and the line UV. Similarly the movement of O" is only within the region limited by n' and n" (see Fig. 4). Consider now the circle T with diameter O'O". Since the slope O'O" (according to (ii)), and the slopes of four arcs at O' and O" (according to (i)) are between ε and -ε, the possible movements of both O' and O" are within the circle T. Clearly any such movement will decrease the distance between O' and O". This is a contradiction, and Theorem 2 is proved. ♦

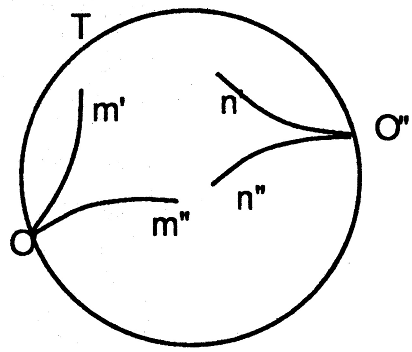

Figure 4

Theorem 3. Four points always suffice to immobilize any tube.

Proof. The idea of the proof is to choose first as immobilizing points two touching points U and V of a diameter circle S and the tube P. Then we show that the only possible motion of the shape P, without one of these two points penetrating the interior of P, is the "sliding" of P between U and V. U and V remain then on the boundary of P. Each point in plane moves along a smooth curve, determined by the "sliding" motion. Two additional points may be chosen, each one to prevent the motion in one of two possible directions of the sliding. This will be easy to do when during such motion some points from the boundary of P move to the interior of P. If this does not happen then it can be proved that no point of P moves to the exterior of P, otherwise P would change its area. In the remaining case, all the points from the boundary of P must remain in the boundary during the motion. Lemma 1 shows that this may happen only when P is a circle. ♦

Lemma 1. Suppose that during the motion of P every point from the boundary of P moves to a point belonging to the boundary of the initial position of P. This is possible only when P is a circle.

Theorem 4. Four points always suffice to immobilize any shape different from a circle.

Proof. Follows from Theorems 1, 2, and 3.♦

Theorem 5. Four points always suffice to immobilize any shape P which has holes (except concentric rings).

Proof. If the boundary of the external shape is a shape which is not a circle, the result follows from Theorem 4. Suppose now that the external boundary is a circle S centered at O. Consider the closest pair of points, one from the external boundary (point U) and one from the boundary of a hole (point V). If |UV| is the unique local minimum, two points U and V suffice to immobilize P (any motion of P must move at least one of U or V, but that point must then penetrate the interior of P). Otherwise in the neighborhood of U and V the shape P is equivalent to a tube, and the proof may be given as in Theorem 3. ♦

3. Conclusions and open problems.

In this paper we studied the problems of immobilization of planar set bounded by a Jordan curve. A number of interesting open problems follow from this work.

An interesting area of further research is an extension to higher dimensions. For shape with holes (ring is excluded), where each hole is bounded by a Jordan curve, we believe that three points should be always sufficient to immobilize it. Moreover, two points should be sufficient in most cases. This is obvious for polygonal shapes where the two points are placed at endpoints of the diameter of a hole.

Finally, we would like to recall three challenging questions asked by Kuperberg which were not addressed in this paper:

- Give an algorithm verifying if a given polygon P is immobilized by a given set I.
- Does there exist a smooth convex shape that cannot be immobilized by three points?
- Say a set C of points not in the interior of P *captures* P if P cannot be moved to infinity without at least one point of C becoming internal to P at some time. Is the minimum number of points needed to capture P always the same as the minimum number of points needed to immobilize it? The answer is negative for general shapes (a shape of the form of letter H is an example) but remains open for convex shapes.

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