ON DELAUNAY AND VORONOI DIAGRAMS OF ORDER K IN THE PLANE

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ABSTRACT.- We define a dual $Del_k(S)$ of the order-k Voronoi diagram $Vor_k(S)$ of a set S of n sites in the plane and call it the Delaunay diagram of order k and we give an iterative algorithm which constructs $Del_k(S)$ in $O(k^2 n \log n)$ time and $O(k(n-k))$ space and we deduce $Vor_k(S)$ from $Del_k(S)$ by a linear duality algorithm .

1.- INTRODUCTION .-

If S is a set of n sites in the Euclidian space E^d of dimension d and if $k \in \{1,\dots,n-1\}$, for every k-subset T of S, the set V(T) of points of E^d which are strictly nearer to all the sites of T than to the sites of $S \setminus T$ is either void or a domain (open and connected) . The nonvoid domains V(T) define the Voronoi diagram $Vor_k(S)$ of order k of S .

Voronoi diagrams have been independently discovered and used in various areas of science. In the plane, the first construction of $Vor_1(S)$ in optimal time $O(n \log n)$ has been given by Shamos and Hoey [SH75] . For $k > 1$, the first construction of $Vor_k(S)$ in the plane is the iterative algorithm of Lee [Le82] and is in $O(k^2 n \log n)$ time and $O(k(n-k))$ space . [EOS86] construct all the $Vor_k(S)$ in optimal $O(n^{d+1})$ time for every dimension d . [Ed86] gives an algorithm which builds $Vor_k(S)$ for a given k in $O(k(n-k)\sqrt{n} \log n)$ time and optimal $O(k(n-k))$ space and [CE87] give an other in $O(k(n-k) \log^2 n + n^2 \log n)$ time and optimal space . Aurenhammer [Au90] has introduced a duality between $Vor_k(S)$ and convex hulls in E^{d+1} and has given an algorithm with the same time complexity as Lee's algorithm and optimal $O(k(n-k))$ space . We do not know of any proof to use the linear algorithm for convex polygons given by [AGSS89] when the selected subset of S is not convex. Lee's and Aurenhammer's algorithms are the most time-efficient for $k < \sqrt{n/\log n}$. A Delaunay tree has been introduced in [BT86] and dynamic constructions of $Vor_k(S)$ are given in [BDT90] .

The main contribution of this paper is the definition of the dual diagram $Del_k(S)$ of $Vor_k(S)$ in the plane and an algorithm which iteratively builds this diagram with the same time complexity and optimal space complexity as Aurenhammer's algorithm. Moreover the same duality algorithm is used to obtain $Vor_k(S)$ from $Del_k(S)$ for every k . We use the algorithm of Elbaz and Spehner [ES90] for the construction of $Del_1(T)$ for some parts T of S. Since this algorithm builds the real Delaunay diagram and not a triangulation which refines it when degenerate cases occur (if more than 3 sites are cocircular) we deal with all the degenerate cases . The angular approximation of this cocircularity introduced by [ES90] is also used and thus, for every similitude $\mathcal{T}$ of the plane E , we have $\mathcal{T}(Del_k(S)) = Del_k(\mathcal{T}(S))$.

2.- THE DELAUNAY DIAGRAM OF ORDER k .-

Definition 1.- Let d be the Euclidian distance in the plane E. For every subset X of E, let $\overline{X}$ be the closure of X and $\delta(X)$ be the boundary of X.

For every k-subset T of S such that $V(T) = \{x \in E \; ; \forall t \in T, \; d(x,t) < d(x, S \setminus T)\} \neq \Phi$, V(T) is called a Voronoi region of order k of S.

For every (k-1)-subset P of S and every sites $s, t \in S \setminus P$, such that $c_P(s,t) = \{x \in E \; ; \forall r \in P, \; d(x,r) < d(x,s) = d(x,t) < d(x, S \setminus (P \cup \{s,t\}))\} \neq \Phi$, $c_P(s,t)$ is called a Voronoi edge of order k of S.

Every remaining point of E is called a Voronoi vertex of order k of S.

The Voronoi regions, edges and vertices of order k of S form a partition of the plane E which is called the Voronoi diagram of order k of S and is denoted by $Vor_k(S)$.

Definition 2.- For every Voronoi region V(T) of order k, the center of gravity g(T) of the set of points T is called a Delaunay vertex of order k of S.

For every Voronoi edge $c_P(s,t)$ of order k of S, $g(P \cup s)$ and $g(P \cup t)$ are Delaunay vertices of order k of S and the open straight-line segment $e_P(s,t)$ between $g(P \cup s)$ and $g(P \cup t)$ is called a Delaunay edge of order k of S.

If S_k and E_k are respectively the sets of Delaunay vertices and Delaunay edges of order k of S, then every domain (or open and connected region) of $E \setminus (S_k \cup E_k)$ is called a Delaunay region of order k of S.

If R_k is the set of Delaunay regions of order k of S, then, by the following theorem, (S_k, E_k, R_k) is a partition of E and this partition is called the Delaunay diagram of order k of S and is denoted by $Del_k(S)$.

Theorem 1.- The sets S_k, E_k and R_k form a partition of E.

Sketch of proof.- (i) If P and Q are k-subsets of S such that $P \neq Q$, $V(P) \neq \Phi$ and $V(Q) \neq \Phi$, $\forall x \in V(P)$ and $\forall y \in V(Q)$ we have

$k\,d(x,g(P))^2 + \Sigma_{r \in P}\, d(g(P),r)^2 < k\,d(x,g(Q))^2 + \Sigma_{s \in Q} d(g(Q),s)^2$ and

$k\,d(y,g(Q))^2 + \Sigma_{s \in Q}\, d(g(Q),s)^2 < k\,d(y,g(P))^2 + \Sigma_{r \in P} d(g(P),s)^2$.

But these two strict inequalities are incompatible with the equality $g(P) = g(Q)$. Thus all the vertices of S_k are distinct.

(ii) If $e_P(s,s')$, $e_Q(t,t') \in E_k$ are such that $\exists\, z \in e_P(s,s') \cap \overline{e_Q(t,t')}$ and thus $\exists\, \alpha > 0$, $\alpha' > 0$, $\beta \geq 0$, $\beta' \geq 0$ such that $\alpha + \alpha' = \beta + \beta' = 1$ and, if $p = \alpha s + \alpha' s'$ and $q = \beta t + \beta' t'$, then $g(P \cup p) = z = g(Q \cup q)$ and this is incompatible with the following strict inequalities : $\forall x \in c_P(s,s')$ and $\forall y \in c_Q(t,t')$,

$k\,d(x,g(P \cup p))^2 + \Sigma_{r \in P \cup p} d(g(P \cup p),r)^2 + \alpha d(p,s)^2 + \alpha' d(p,s')^2$

$< k\,d(x,g(Q \cup q))^2 + \Sigma_{r \in Q \cup q} d(g(Q \cup q),r)^2 + \beta d(q,t)^2 + \beta' d(q,t')^2$

and $k\,d(y,g(Q \cup q))^2 + \Sigma_{r \in Q \cup q} d(g(Q \cup q),r)^2 + \beta d(q,t)^2 + \beta' d(q,t')^2$

$< k\,d(y,g(P \cup p))^2 + \Sigma_{r \in P \cup p} d(g(P \cup p),r)^2 + \alpha d(p,s)^2 + \alpha' d(p,s')^2$.

It follows that the edges of E_k do not cut and do not contain vertices of S_k. Since by definition, the Delaunay regions are disjoint, (S_k, E_k, R_k) is a partition of E.

Theorem 2.- There exists an orthogonal duality between the diagrams $Del_k(S)$ and $Vor_k(S)$.

Sketch of proof.- Every Voronoi region $V(T)$ of $Vor_k(S)$ is associated with the vertex $g(T)$ of S_k. Every Voronoi edge $c_P(s,t)$ is included in the bisector of (s,t) and is associated with the Delaunay edge $e_P(s,t)$ which is parallel to the straight line st and hence perpendicular to $c_P(s,t)$. For every Voronoi vertex u, there exists a p-subset P of S such that the circle $K_P(u)$ with centre u and radius $d(u, S \setminus P)$ contains a circular sequence $(s_1,\ldots,s_q)$ of sites of $S \setminus P$ such that $p < k < p+q$ then u is associated with the Delaunay region $R(u)$ whose successive vertices are $g(P \cup \{s_i,\ldots,s_{i+k-p-1}\})$ with $i \in \{1,\ldots,q\}$.

Definition 3.- Let P be a $(k-1)$-subset of S such that $g(P) \in S_{k-1}$.

Every Delaunay vertex $g(P \cup s)$ of S_k is said to be attached to P. Let $\Phi(P)$ be the set of Delaunay vertices which are attached to P.

Every Delaunay edge $e_P(s,t)$ of E_k is said to be attached to P.

Every Delaunay region $R(u)$ of R_k every edges of which are attached to P is said to be attached to P. If r is the number of vertices of $\delta(R(u))$, $rem_k(R(u)) = r-2$ is called the remanence degree of $R(u)$.

A region $R(u)$ of R_k which is not attached to some P is said to be neutral. Then there exists a region $R'(u)$ of R_{k-1} and we define $rem_k(R(u)) = rem_{k-1}(R'(u)) - 1$.

The union of the Delaunay vertices, edges and regions which are attached to P is called the territory of P and is denoted by $ter(P)$.

We give the following theorem without proof.

Theorem 3.- For every $(k-1)$-subset P of S such that $V(P) \neq \Phi$, $Del_k(S)$ and $Del_1(\Phi(P))$ have the same trace on $ter(P)$.

3.- BUILDING THE DELAUNAY DIAGRAM OF ORDER k.-

First we use the algorithm of [ES 90] to construct $Del_1(S)$ and then, for every $k > 1$, we derive $Del_k(S)$ from $Del_{k-1}(S)$ by the following algorithm :

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delaunay(S,k);
    for every vertex g(P) of S_{k-1} do territory(g(P));
    for every region R(u) of R_{k-1} such that rem_{k-1}(R(u))> 0 do assembly(R(u));
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where the procedures territory and assembly are

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territory(g(P));
    step 1 : Ψ(P) := Φ; U(P) := Φ ; for every two successive neighbors g(P ∪t\ s)
          and g(P ∪t'\s') of g(P)such that s,s'∈ P and t ≠ t'do
          { Ψ(P) := Ψ(P)∪t ; U(P) := U(P)∪triangle(g(P),t,t') }
    step 2 : compute Del_1(Ψ(P)) ;
    step 3 : compute the restriction of Del_1(Ψ(P)) to U(P)∩ convexhull(Ψ(P));
    step 4 : compute the image ter(P) of this diagram by the homothety centered
             in g(P) and with factor 1/k ;
          for every Delaunay region R attached to P determine rem_k(R);
assembly(R(u));
    determine the region R'(u) of R_k ; rem_k(R'(u)):= rem_{k-1}(R(u))- 1;
    for every  edge ep(s,t) of R'(u) stick ter(P) on R'(u) along ep(s,t);
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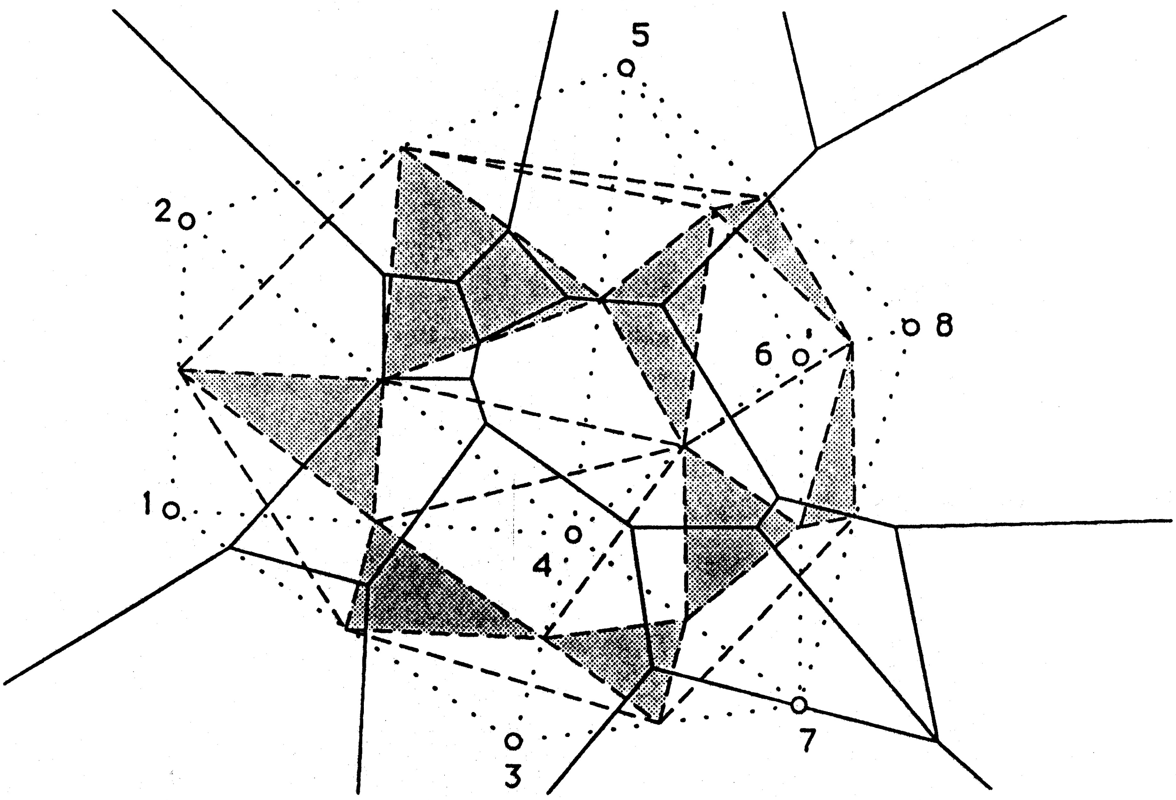
Theorem 4.- This algorithm builds $Del_k(S)$ in $O(k^2 n \log n)$ time and $O(k(n-k))$ space.

Sketch of proof.- By theorem 3, $Del_k(S)$ is correctly built. If $n_{k-1} = |S_{k-1}|$ steps 1, 3 and 4 are in $O(n_{k-1})$ and step 2 is in $O(n_{k-1} \log n)$ since $\Phi(P) \subset S$ and every edge of E_{k-1} is used twice. Lee has proved that $n_k \in O(k(n-k))$. It follows that the iterative algorithm which builds $Del_k(S)$ is in $O(k^2 n \log n)$ time and $O(k(n-k))$ space.

Theorem 5.- $Vor_k(S)$ is obtained from $Del_k(S)$ in optimal $O(k(n-k))$ time and space.

4.- CONCLUSION.- We have introduced the notion of Delaunay diagram only in the plane, but it is not difficult to define $Del_k(S)$ and to prove the orthogonal duality with $Vor_k(S)$ when $d > 2$. The algorithm which builds $Del_k(S)$ can also be generalized.



$Del_1(S)$ (dotted lines), $Del_2(S)$ (dashed lines) and $Vor_2(S)$ (solid lines); 3,4,6,8 are cocircular and neutral regions of $Del_2(S)$ are in grey.

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