BENDING AND STRETCHING ORDERS INTO THREE CHANNELS

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The chief graphical data structure for an ordered set is its **diagram**. Accordingly the elements of the ordered set P are drawn on the plane as disjoint small circles, arranged in such a way that, for a and b in P, the circle corresponding to a is higher than the circle corresponding to b whenever a > b and a monotonic arc is drawn to connect them just if a covers b (that is, for each $x \in P$, $a > x \geq b$ implies $x = b$). (We also say that a is an upper cover of b or b is a lower cover of a.) Usually it is convenient to use straight segments for the edges.

We are motivated by the problem to represent an ordered set graphically on an output tape of arbitrary length and bounded width. We suppose that we can print the circles representing the points along a (small) number of vertical "channels" (chains). In a finite ordered set, the minimum number of chains whose union is all of the set equals the maximum size of an antichain [Dilworth (1950)]. In graphical terms it is convenient to visualize chains as "vertical paths", using vertical line segments for the edges. Of course, it is not possible to draw vertically all chains on the plane - unless, for every vertex, both the down degree (that is, the number of lower covers) and the up degree (the number of upper covers) is at most one. Occasionally it may be convenient to draw the disjoint chains of a given chain decomposition vertically, using vertical line segments for all edges in each of its chains. In this case we call the drawing a **vertical diagram** - or a **k-vertical diagram**, if it has k disjoint, vertical chains.

What we are after is a vertical diagram, as a "grid", into which to embed the diagram of the ordered set.

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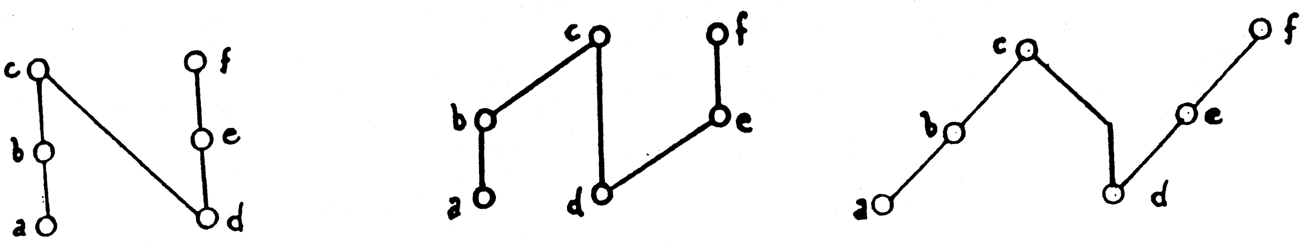


A 2-vertical diagram A 3-vertical diagram Not a vertical diagram

An order P = {a,b,c,d,e} with chain decomposition {a,b,c} and {d,e,f}.

Figure 1

What is the smallest number of channels needed?

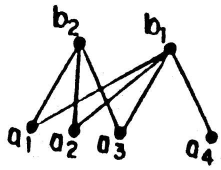
To be more precise, for an ordered set and a positive integer k, a k-**channel diagram** is a diagram of it as a subdiagram of a k-vertical diagram. There are obvious applications of such k-channel diagrams to "drawing" and to "reading" pictures of sets of jobs, subject to precedence constraints, in human decision-making problems, as well as to the automation of such diagrams. These diagrams may also be applied to circuit design, multi-processor scheduling using queues, and even to a distributed computing environment for the processing of an ordered set of tasks.

Here are our main results.

THEOREM (i) Bending and stretching the edges of the diagram of an arbitrary n-element ordered set will transform it into a 3-channel diagram.

(ii) A 3-channel diagram can be drawn dynamically in time $O(n^2)$ on a 3 by $3n^2$ grid.

(iii) There are finite ordered sets with no 2-channel diagram. Moreover, there are finite ordered sets which cannot be embedded into ordered sets which, with bending and stretching, have a 2-channel diagram.



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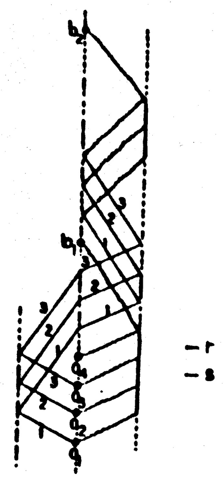


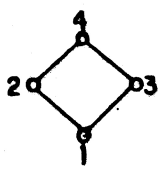
Bending and stretching P
into a 3-channel
embedding.

Figure 2

The representation ensured by this construction has several striking features. Although the 3-channel diagram of the initial order uses monotonic arcs (piecewise straight) for its edges, the 3-vertical diagram which contains it uses only straight segments for its edges.

Another feature of the construction is that all of the vertices of the ordered set can be placed, according to any linear extension at all, along a vertical, with no edges drawn along this vertical. This, in turn, recalls the idea of a "book embedding", a particular rendering of the diagram of an ordered set according to which the vertices are placed along a "spine" (vertical) and the edges are placed on "pages" (infinite half planes) so that each edge lies on only one page and no edges cross. There are ordered sets which require many pages [Nowakowski and Parker (1989)]. On the other hand, any order (and any graph) has a subdivision which requires at most three pages (cf. [Bernhart and Kainen (1979)]). If pages are replaced by surfaces of cylinders then, according to the construction used to prove our Theorem, every ordered set can be represented using just two "cylinder pages" (see Figure 3).

COROLLARY. **The diagram of any finite ordered set can be drawn on the surface formed of two upright cylinders with a common tangent in such a way that all of the diagram vertices are located on this common tangent while the edges are drawn on the surface as non-crossing, monotonic arcs.**



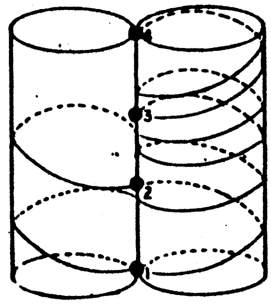


Figure 3

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