Detecting and Computing Intersections of Convex Chains

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ABSTRACT

Let $P_c = [p_1,p_2,...,p_r]$ and $Q_c = [q_1,q_2,...,q_s]$, $r+s=n$, be two convex polygonal chains in the plane. We investigate the problem of detecting whether these chains intersect and computing their intersection if it exists. Along the way, we characterize when all the intersections can be reported in O(log n) time and when the detection problem has complexity $\Omega(n)$.

1. Introduction

Detecting and computing intersections of convex objects is a fundamental operation in a number of applications areas such as graphics, robotics, VSLI, automated cartography and geographic information systems [CD 87]. Let P = $[p_1,p_2,...,p_r]$ and Q = $[q_1,q_2,...,q_s]$, $r+s=n$, be two convex polygons in the plane. Then it is known that $P \cap Q$ is a convex polygon that may contain O(n) vertices and several algorithms exist for computing $P \cap Q$ in O(n) time [SH 76, OCON 82, To 85]. Futhermore, if it is only required to know whether or not the two polygons intersect (the *detection* problem) then if the polygons are represented with data structures that afford binary search, O(log n) time suffices to detect if P and Q intersect and to provide a witness if the answer is in the affirmative [CD 80, CD 87]. On the other hand, to detect whether one polygon lies in the interior of the other has complexity $\Omega(n)$ [CD87].

In this paper we consider the hitherto unexplored area of detecting and computing intersections among a pair of convex polygonal chains.

2. Convex Chains

Let $P_c = [p_1,p_2,...,p_r]$ and $Q_c = [q_1,q_2,...,q_s]$, $r+s=n$ be two convex polygonal chains in the plane. We assume that the polygonal chains are represented by arrays with their vertices given in counterclockwise order and that no three vertices of a chain are colinear. A polygonal chain $P_c = [p_1,p_2,...,p_r]$ is convex if for every three consecutive vertices $p_i,p_j,p_k \in P_c$, i= 1,2,...,r-2, p_k is to the left of $L(p_i,p_j)$, the line containing p_i and p_j. Let θ_j be the left angle that the line $L(p_i,p_j)$ makes with $L(p_j,p_k)$, for j

= 2,...,r-1 as shown in figure 1. We use θ_j to

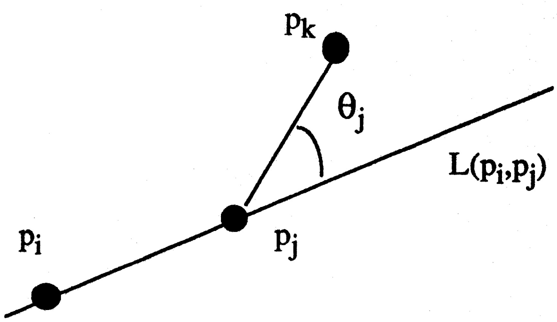


Figure 1 The angle θ_j

associate an angle $A^P_{i,i+1}$ with each segment $[p_i,p_{i+1}]$ of P_c as follows. If the origin is located at p_1, $A^P_{1,2}$ is the minimum counterclockwise angle necessary to rotate the positive x axis to align with the ray from p_1 through p_2. For i = 2 , ..., r-1, $A^P_{i,i+1} = A^P_{i-1,i} + \theta_i$. Note that, if the origin is located at p_i, $A^P_{i,i+1}$ mod 360 is the minimum counterclockwise angle necessary to rotate the positive x axis to align with the ray from p_i through p_{i+1}. We define the angle interval of P_c, AI^P, to be the real interval $[A^P_{12}, A^P_{r-1,r}]$.

Although detecting intersection between an arbitrary pair of convex chains has an $\Omega(n)$ lower bound (see section 5), if restrictions are put on the angle intervals of the chains this lower bound may no longer apply.

Without loss of generality let us assume we are dealing with two convex chains P_c and Q_c such that $A^Q_{1,2} = 0°$. If $(AI^P \bmod 360) \cap (AI^Q \bmod 360) = \varnothing$, we say that P_c and Q_c are angle separable. That is , $P_c$and Q_c are angle separable if and only if $A^Q_{S-1,S} < A^P_{1,2}$ and $A^P_{r-1,r} < 360°$. We say two angle separable polygonal chains, P_c and Q_c, are a bounded pair if $A^P_{r-1,r} < 180°$. That is, $P_c$and Q_c are a bounded pair if and only if $AI^P$and AI^Q do not intersect and both AI^P and AI^Q are contained in the interval [0, 180).

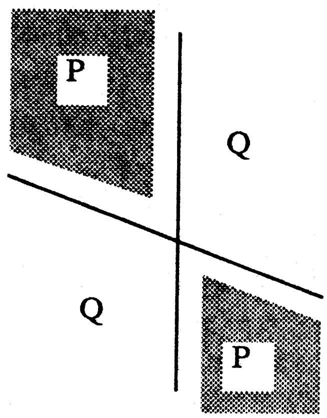
3. Bounded Pairs of Chains

Angle separable bounded pairs of chains arise naturally in the solution to some geometric problems such as determining the envelope of a set of lines [Ke 91]. Throughout this section we will assume that P_c and Q_c are an angle separable bounded pair of convex polygonal chains.

Let L_1 and L_2 be the lines through the origin that form the angles A^P_{12} and $A^P_{r-1,r}$, respectively, with the positive x axis. These two lines divide the plane into four regions. If some point of P_c lies at the origin then the rest of P_c lies in the two opposing regions defined by L_1 and L_2 that would be swept first if L_1 were rotated counterclockwise. If $A^P_{12} = 90°$ and $A^P_{r-1,r} = 160°$ we have the situation illustrated in figure 2.

Figure 2

Regions containing a bounded pair of angle separable chains P_c and Q_c



If Q_c intersects P_c at the origin then the rest of Q_c must lie in the two opposing regions defined by L_1 and L_2 which are not occupied by P_c. This fact that the rest of the chains lie in disjoint regions implies the following lemma.

Lemma: An angle separable bounded pair of convex polygonal chains intersect in at most one point.

The properties of angle separable bounded pairs of convex chains allow us to find this intersection (or determine that it does not exist) efficiently.

Theorem: Given an angle separable bounded pair of convex polygonal chains $P_c = [p_1,\ldots,p_r]$ and $Q_c = [q_1,\ldots,q_s]$, $r+s = n$, then in O(logn) time an algorithm can detect whether the chains intersect and if they do determine the intersection point.

Proof: Let $m_P = p_{\lfloor r/2 \rfloor}$ be the median vertex of the chain P_c. Likewise let m_Q be the median vertex of the chain Q_c. Let L_1 and L_2 be the lines that pass through m_P that form the angles $A^P{}_{12}$ and $A^P{}_{r-1,r}$, respectively, with the line through m_P parallel to the x axis. As we have already seen L_1 and L_2 divide the plane into four regions, two of which contain part of P_c. Depending upon which of these four regions contains the point m_Q we can discard either half of P_c or half of Q_c as not possibly intersecting the other chain. Two of the four cases are shown in figure 3. In case 1 the left half of Q_c can be discarded. In case 2 the upper half of P_c can be discarded. Since in constant time one half of one chain is eliminated the algorithm will reduce one chain to a single segment in O(logn) time. The intersection of the single edge with the remaining chain can be completed in O(log n) time using binary search [CD 80].

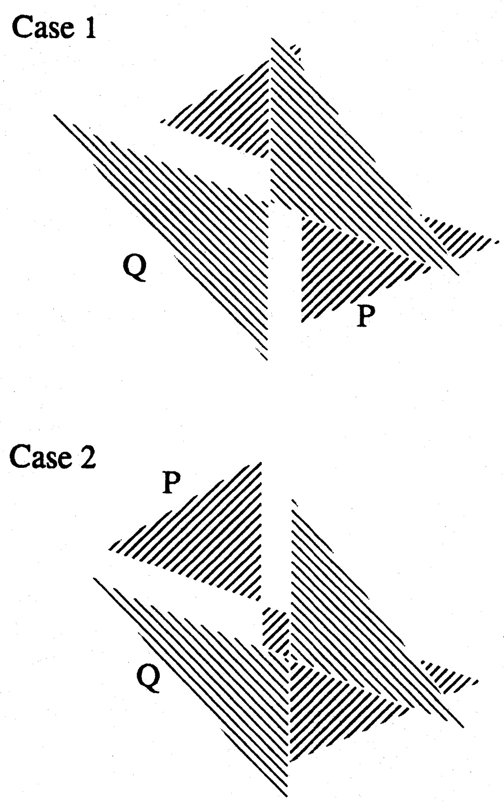


Figure 3

4. Angle Separable Chains

If two angle separable chains do not form a bounded pair, the chains intersect at most twice and we are able to prove the following lemma in the full paper [KT 91].

Lemma: If two angle separable chains, P_c and Q_c, intersect, then they intersect in either a point, a line segment, or in two distinct points.

We are also again able to develop an algorithm to find these intersections efficiently. This algorithm is more complex than for the case of a bounded pair of chains. Nevertheless, the same case based approach of eliminating half of one of the chains lies at the heart of the algorithm. The following theorem is proven in the full paper [KT 91].

Theorem: All intersection points between two angle separable polygonal chains, P_c and Q_c, can be computed in O(log n) time, where n = r + s.

5. Angle Overlapping Chains

If two convex polygonal chains are not angle separable we have the following theorem

Theorem: Given $\varepsilon > 0$, there is a lower bound of $\Omega(n)$ on the time to detect whether two convex polygonal chains P_c and Q_c intersect even when $(AI^P \bmod 360) \cap (AI^Q \bmod 360) = [A_1,A_2]$ and $A_2 - A_1 \leq \varepsilon$.

Proof: We construct P_c and Q_c using polar coordinates. Let r be a positive number. Let $P_c = [p_0,\ldots,p_n]$ where $p_i = (r, 270° + i\varepsilon/n)$, for i = 0,...,n. Let $r' = r\cos(\varepsilon/2n)$ and let $Q_c = [q_0,\ldots,q_n]$ where $q_i = (r', 270° + i\varepsilon/n)$ for i = 0,...,n. We have then that $AI^P = AI^Q = [\varepsilon/2n, \varepsilon - \varepsilon/2n] = [A_1,A_2]$, thus $A_2 - A_1 = \varepsilon - \varepsilon/2n - \varepsilon/2n < \varepsilon$. Furthermore for i = 2,..., n-2, vertex q_i of Q_c can be translated a suitable distance away from the origin such that an intersection is created between P_c and Q_c without altering any other vertex and without changing AI^P or AI^Q. Therefore at least n-4 of the vertices of Q_c must be checked to detect an intersection.

6. Conclusions

Given two convex polygonal chains in the plane, we have characterized, in terms of angle separability, when an intersection between the chains can be efficiently detected and computed. In particular, if the two chains are angle separable all intersections between the chains can be computed in O(log n) time, whereas, if they are not angle separable an $\Omega(n)$ lower bound applies, even to detecting whether they intersect.

7. Acknowledgements

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