Extended Abstract:

Offsets of Circular Arc Figures

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Abstract

A simple plane sweep algorithm is presented for computing the "offset" of "circular arc figures". (The "offset" operation is important in solid modelling. "Circular arc figures" are planar sets whose boundaries consist of a finite number of circular arcs and/or line segments forming a set of disjoint Jordan curves; the figures need not be simply connected.) The algorithm runs in $O(n \log^2 n)$ time, where n is the number of arcs and line segments in the boundary of the circular arc figure. To analyse the time complexity, a general topological result of interest in its own right is proved: given what we call a *k–admissible* collection of m Jordan curves, the number of curve intersection points that appear on the boundary of the union of the regions bounded by these curves is at most $k(3m-6)$, a tight bound. This generalizes a result of Kedem, Livne, Pach and Sharir.

1. Introduction

We have have obtained a simple plane-sweep algorithm for computing "offsets" of "circular arc figures", together with a running time analysis of the algorithm. (The quoted terms in this sub-section are defined later in the Introduction.)

To carry out the running time analysis, we have proven a general topological result of interest in its own right. This result gives a tight upper bound for the number of curve intersection points that appear on the boundary of the union of certain regions bounded by Jordan curves (these Jordan curves are not necessarily polygonal.) Our result generalizes work of Kedem, Livne, Pach

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and Sharir [KLPS]. (Their notion of an "admissible" collection of planar Jordan curves is what we call a "2-admissible" collection, and we have generalized their results to what we call "k-admissible" collections, for non-negative even integers k.)

Let S be a set in the plane. The l *expansion offset* $E(l, S)$ of S is defined as the Minkowski sum of S with a closed disc of radius l centered at the origin (see, for example, [SPD]). That is, $E(l, S)$ is the union of the closed discs of radius l with centers in S. There is another type of offset called the *shrink* offset [SPD]. Our paper considers exclusively expansion offsets, which we simply call offsets.

The computation of the "offset" of a set is an important operation in geometric modeling about which surprisingly little has been published. It has applications in tolerance analysis, clearance testing, NC (numerical control) code generation, finite element mesh generation, etc., as pointed out in [RR] and [SPD]. Farouki [Far] outlined exact offset procedures for convex polyhedra, convex solids of revolution, and convex solids of linear extrusion. Rossignac and Requicha [RR] described offset operations for solids of simple structure that are represented both in constructive solid geometry form and in boundary representation form.

Our interest in computing the offsets of circular arc figures came about in the process of designing a brute-force algorithm for the Euclidean Tree Placement problem. The Euclidean Tree Placement Problem, informally speaking, is to position in the plane a tree whose edges have positive integer weights in such a way that the distance between each pair of adjacent nodes is equal to the weight of the edge between them and in such a way that any pre-assigned positions to particular nodes are respected. This problem finds application in the design of grasp positions for robot hands and in computer animation. Our algorithm for computing offsets of circular arc figures can likely be extended to the computation of offsets of other kinds of figures.

- *Other Related Work*

It should be noted that our notion of "circular arc figure" differs from the notion of "splinegon" of Souvaine [Souv].

Other related work includes [BO], [EGHPPSSS], [OWW], and [SH].

- *Basic Definitions*

We call a planar set F a *figure* if F is connected and compact and if its boundary points form a set of disjoint Jordan curves. We call F a *circular arc figure* if its boundary consists of a finite set of circular arcs (of various radii). We allow the radius of a boundary arc to be infinite, so line segments can appear in the boundary. Let $J = \{ J_i \mid i = 1, \ldots, m\}$ be a collection of Jordan curves in the plane. Let $int(J_i)$ be the interior of Jordan curve J_i. Let $K(J_i)$ be the closure of $int(J_i)$ (the union of $int(J_i)$ and J_i). Let k be a non-negative even integer. We define J to be *k-admissible* if: (1) for each $i \neq j$, the intersection $J_i \cap J_j$ consists of an even number of crossing points not exceeding k; and (2) for each $i \neq j$, $K(J_i) - int(J_j)$ is connected.

In [KLPS], a collection of planar Jordan curves is called *admissible* if the intersection of every two curves either consists of two crossing points or is empty. Thus, an admissible Jordan curve collection in that paper is just what we call *2-admissible*. They also call a collection of planar Jordan curves *weakly admissible* if the intersection of every pair of them consists of at most two points.

- *Main Results*

In this paper, we present a simple $O(n \log^2 n)$ time algorithm for computing the expansion offset of a circular arc figure bounded by n arcs. To do the running time analysis of the algorithm, we prove the following topological result. For a k-admissible collection J of m Jordan curves, $J = \{ J_i \mid i = 1, ..., m\}$, the number of the intersection points of the J_i that appear on the boundary of $\bigcup_i K(J_i)$ is bounded by $k(3m-6)$ for all $m \geq 3$. This bound is tight. Moreover, this result can be further generalized to what we call *k–weakly–admissible* collections of Jordan curves.

2. The Algorithm

The following notation will be used:

NOTATION:

$\partial(F)$ — the boundary of figure F.

Δ_x^l — the closed disc of radius l centered at x.

D_x^l — the open disc of radius l centered at x.

$\delta(x, S)$— $\min_{y \in S} \delta(x, y)$, where $\delta(x, y)$ is the distance between points x and y, and S is a closed set.

Lemma 2.1. Let F be a circular arc figure. Then the boundary of $E(l, F) = \bigcup_{x \in F} \Delta_x^l$ is contained in the boundary of $E(l, \partial(F)) = \bigcup_{x \in \partial(F)} \Delta_x^l$.

Proof. Suppose that p is a point on the boundary of $E(l, F)$. The disc Δ_p^l must *touch* a point $q \in \partial(F)$. Therefore p is contained in $E(l, \partial(F))$. If p is not contained in the boundary of $E(l, \partial(F))$, then there would exist an open disc D_p^ε for some $\varepsilon > 0$ such that D_p^ε is completely within $E(l, \partial(F))$. Because $\partial(F) \subseteq F$, D_p^ε would also be completely contained in $E(l, F)$. This contradicts the assumption that p is a boundary point of $E(l, F)$. □

Lemma 2.2. Let F be a circular arc figure. Let p be a point on the boundary of $E(l, \partial(F))$. Then p is a boundary point of $E(l, F)$ if and only if p is not contained in F.

Proof. It is easy to see that if p is a boundary point of $E(l, F)$, then p is not contained in F.

Now assume p is not contained in F. We show that for any $\varepsilon > 0$, there exist points $x, y \in D_p^\varepsilon$ such that $x \in E(l, F)$ and $y \notin E(l, F)$. Since $p \in \partial(E(l, \partial(F)))$ (by the boundary point definition), for any $\varepsilon > 0$, there exist points $x, y \in D_p^\varepsilon$ such that $x, y \notin F$, $x \in E(l, \partial(F))$ and $y \notin E(l, \partial(F))$. Clearly, $x \in E(l, F)$ because $\partial(F) \subseteq F$. For point y, there are two cases: (1) $\delta(y, F) > l$, and (2) $\delta(y, F) < l$. ($\delta(y, F)$ cannot be exactly l because $y \notin E(l, \partial(F))$.) If (1), then $y \notin E(l, F)$. If (2), then Δ_y^l must intersect F or completely contain F. If Δ_y^l intersects F, then $y \in E(l, \partial(F))$. This contradicts the assumption for y. If Δ_y^l contains F, then $\delta(y, F) < l$, thus $y \in E(l, \partial(F))$. The same contradiction arises. □

The above two lemmas remain true for some more general figures.

- *Outline of the algorithm*

For what follows, we remark that it is straightforward to show that the offset of a single circular arc (i.e., just the simple 1-dimensional set consisting of an arc of a cirle) is a solid circular arc *figure* bounded by at most *four* arcs.

An outline of our algorithm for the offset computation of a circular arc figure F is as follows. First, we compute the boundary ∂_i of $E(l, \alpha_i)$ for every arc $\alpha_i \subseteq \partial(F)$. Then we compute the boundary of $\bigcup_i K(\partial_i) = E(l, \partial(F))$. However, as shown in Lemma 2.1, the boundary of $E(l, \partial(F))$ is a superset of the boundary of $E(l, F)$, and hence may contain *inappropriate* arcs which do not belong to $\partial(E(l, F))$. Our final step is to eliminate those inappropriate arcs.

3. K-admissible Collections of Jordan Curves

Our tight upper bound on the number of intersection points that appear on the boundary of the union of the regions bounded by the Jordan curves in a k-admissible collection requires many technical lemmas and involves an inductive argument indexed by a triple of parameters defined by how the curves intersect each other. The details appear in [Zhao].

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