MINIMUM COVERS FOR GRIDS AND ORTHOGONAL POLYGONS by PERISCOPE GUARDS (Extended Abstract)

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Abstract: In this paper we try to determine the complexity of finding guard covers in orthogonal polygons by considering periscope visibility. We show that finding minimum periscope (as well as k-bend and s-guard) covers is NP-hard for 3-d grids. We present an $O(n^3)$ time algorithm for finding minimum periscope guard covers in a simple grid with n segments. We also show that this result can be used to obtain minimum periscope guard covers for a class of simple orthogonal polygons in $O(n^3)$ time.

Introduction: The problem of covering a polygon with the minimum number of star polygons has attracted the interest of many researchers [O'R87,ST88]. This problem is equivalent to the problem of placing minimum number of point guards (minimum guard cover) so that each point inside the polygon is visible to some guard. This problem is shown to be NP-hard [O'R87]. However, the complexity of finding minimum guard cover for orthogonal polygons is open [ST88]. In the standard definition of visibility two points are said to be visible if the straight line joining them does not intersect the exterior of the polygon. Alternative definitions of visibility have been considered for orthogonal polygons [K86,MRS88]. Two points inside an orthogonal polygon are said to be **s-visible** [MRS88] if they can be joined by an orthogonal convex staircase path that does not intersect the exterior of the polygon. Two points are said to be **r-visible** [K86] if they can be placed inside an orthogonal rectangle that is completely contained in the polygon. The notion of r-visibility and s-visibility directly leads to **s-star** and **r-star polygons**. Two points are visible under **periscope visibility** if there is an orthogonal path with at most one bend connecting them without intersecting the exterior of the polygon. Generalizing, **k-bend visibility** allows staircase paths with at most k bends (periscope visibility is the same as 1-bend visibility). If k can have any value but the paths are restricted to be orthogonally convex we have **s-visibility**.

II. Periscope Guard Covers for Grids: The complete two dimensional grid of size n is the graph with vertex set $V = \{1,2,...,n\}X\{1,2,...,n\}$ and edge set $E = \{\{(i,j),(k,m)\} : |i-k|+|j-m| = 1\}$. The complete 3-d grid is defined similarly. A (partial) grid is any subgraph of the complete grid. In a geometric setting we think of the grid edges as corridors and the grid vertices as interconnections of corridors. We also assume that the grid edges are parallel to the major axes. Finding a minimum set of guards needed to cover (under normal visibility) a 3-d grid is NP-hard [N86]. The reduction is from the vertex cover problem for graphs with maximum degree three [O'R87]. We use a similar approach to establish the following theorem.

Theorem 1: Finding the minimum number of periscope guards needed to cover a 3-d grid is NP-hard. (Proofs are omitted in this extended abstract due to lack of space)

Corollary 1: The minimum cover problems for k-bend guards and s-guards in 3-d grids are NP-hard.

We now proceed to develop an $O(n^3)$ time algorithm for minimally covering a 2-d grid by periscope guards. A **grid segment** is any maximal straight line sequence of successive grid edges.

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* Partially supported by a grant from Texas Instruments, Inc.
# Partially supported by a grant from Army Research Office.

A grid is called a **simple grid** if all the end points of its segments lie on the outer face of the planar subdivision formed by the grids (as in Figure 1a); otherwise, the grid is called a **general grid** (a general grid may have holes as in Figure 1b). The crossing set C_i of a segment s_i is the set of segments that intersects s_i. A segment s_1 is said to be **dominated** by a segment s_2 if C_1 is a subset of C_2. In Fig. 1a segments c, e, g and n are dominated by segment d. A segment s is called a **cross** if there exist a segment s_1 such that the crossing set of s_1 contains only s. A segment is called a **pseudo cross** if it becomes a cross by removing zero or more segments dominated by it (note that every cross is also a pseudo cross). Two segments are **equivalent** if their crossing sets are the same. In figure 1a, segments 4, 5 are crosses, segment d is a pseudo cross, and segments c, e are equivalent. Note that a periscope guard that can see a segment s can see all segments equivalent to s. Therefore we keep only one segment from each set of equivalent segments. The importance of domination is illustrated in Figure 1a. A guard located on a dominated segment can be moved to the dominating segment and still see all segments visible from its original position (as well as some additional ones). For example, a guard at point x can be moved to point y in Figure 1a and still see all the segments visible from x. This indicates that certain segments are more important than others. We capture this idea by defining a reduced grid. Let G be a simple grid. Mark all segments that are dominated in G. The grid obtained by removing all marked segments is called the **reduced grid**. Figure 2 shows a grid with the dominated segments marked and the reduced grid obtained by removing them.

Lemma 1: The reduced grid of any simple and connected grid is simple and connected.

Let $C = \{s_1, s_2, ..., s_k\}$ be the crossing set of a segment s in the reduced grid. The crossing set C is said to form a **group** if there exist a segment $s' \in C$ such that s' is dominated by all segments in C. Then s' is called a **junior segment** in C. In Figure 2, the crossing set for segment d forms a group and segment 6 is a junior segment in this group. Note that junior segments are not unique.

Let R, R' be two guard covers for a simple grid. We say that a guard g_i in R is **equivalent** to a guard g_j in R' if the two guards see exactly the same set of grid segments. A guard g_i **covers** a guard g_j if the set of segments visible to g_i contains the set of segments visible to g_j.

Lemma 2: Let G_r be the reduced grid of a simple grid G. Let s be a pseudo cross segment in G_r such that its crossing set $C = \{ s_1, s_2, \dots , s_k \}$ forms a group. Let $s_i \in C$ be a junior segment in the group. Then there exists an optimum guard cover for G that contains a guard equivalent to a guard placed at the intersection of s and s_i.

Lemma 3: The reduced grid G_r of any simple grid G contains a segment whose crossing set forms a group.

Our approach for finding a minimum guard cover for a simple grid is to identify places where any optimum solution should have a guard, place a guard, remove a portion of the grid and repeat until all of the grid is visible. To obtain a minimum guard cover we locate each guard so that any minimum guard cover will contain a guard equivalent to it or covered by it. Dominated segments in the given grid G are marked and a reduced grid G_r is obtained from G by removing dominated segments. Lemma 3 guarantees that at least one segment of G_r is such that its crossing set forms a group. Once such a segment is found, the location of a guard g that corresponds to an optimum solution is determined by using Lemma 2. The details of the algorithm is omitted in this extended abstract and will be reported in the full paper.

Theorem 2: A minimum periscope guard cover for a simple grid can be found in $O(n^3)$ time.

The above idea of identifying groups for placing guards does not work for general 2-d grids. It is easy to construct a 2-d grid in which no crossing set forms a group (Figure 3).

III. Periscope Guard Covers for Simple Orthogonal Polygons: Consider the subdivision formed by extending the edges of an orthogonal polygon into its interior. The polygon now consists of rows and column of rectangles. We construct a grid G to represent an orthogonal polygon P by associating a grid segment with each sequence of rectangles. Then the internal grid vertices represent individual rectangles in the polygon (Figure 4a).

Lemma 4: The grid G is simple and connected grid.

Lemma 5: Let X be a guard cover for the grid G. Then X is also a guard cover for the underlying polygon P.

The difficulty with grid G is that the reverse of Lemma 5 is not true. A guard cover for the polygon does not always correspond to a guard cover in the grid. We say that a grid segment is a **swept segment** if there is a grid segment that intersects it and it can be moved through the entire span of the swept segment without intersecting the exterior of the polygon and while both its ends remain in contact with the boundary of the polygon. This definition can be applied recursively by removing swept rectangles from P each time. All but the bottom horizontal segment in Figure 5a are swept (recursively). Replacing corresponding segments in the grid with their intersections with the sweeping segment results in the grids of Figure 5b. Note that a single guard can cover each of the grids. We call the grid resulting from sweeping the **swept grid** G_s for the orthogonal polygon. There is one more problem with both grids G and G_s. It arises when the polygon contains corners like the one shown in Figure 6. There are two orthogonal grid segments that enter the corner and a guard placed on either one of them (i.e., a or b) can see the whole corner. This is similar to what happens when we have swept segments but here we have a choice of two ways to sweep. The problem is that it is not clear locally which of the two choices is best. We refer to this type of corner as a **swept corner**. Note that choosing to sweep with a vertical (respectively horizontal) segment is equivalent to adding a vertical (horizontal) notch into the corner so that the notch is not visible from the vertical (horizontal) edge that enters the corner. Also, note that addition of such a notch eliminates one of the choices, i.e., the grid cover corresponds to a polygon cover.

Lemma 6: A grid cover in the grid G obtained from a simple polygon without swept corners after replacing swept segments with points is equivalent to a polygon guard cover.

Theorem 3: A minimum periscope guard cover for a simple orthogonal polygon with a fixed number of swept corners can be constructed in $O(n^3)$ time.

IV. Concluding Remarks: We presented an $O(n^3)$ algorithm for finding optimum periscope guard cover for simple grids and orthogonal polygons with a constant number of corners. There are many interesting open problems. These include the periscope cover problem for general 2-d grids, the k-bend guard cover problem for grids and orthogonal polygons. Our motivation for considering periscope guards is to help determine the complexity of guard cover problem for orthogonal polygons which remains a well known open problem.

An $O(n^{10})$ time algorithm for minimally covering an orthogonal polygon under s-visibility by using perfect graph approach is reported in [MRS88]. It is tempting to use this approach under periscope visibility also. However the complexity remains the same and hence is too expensive.

References

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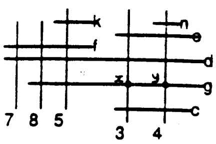

Figure 1a: A Simple Grid.

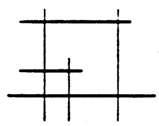

Figure 1b: A General Grid.

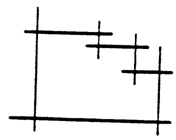

Figure 3: A General Grid whose no crossing set forms a group

(a): Grid

(b): Reduced Grid

Figure 2: Illustrating the construction of Reduced Grid

(a)

(b)

Figure 4: Simple Grid of an Orthogonal Polygon

(a): Grid

(b): Swept Grid

Figure 5: Showing the Construction of Swept Grid

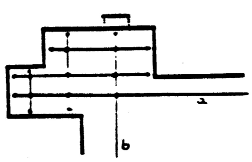

Figure 6: A Swept Corner in an Orthogonal Polygon