Towards a General Theory of Visibility*

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Abstract

Many notions of convexity have been introduced in computational geometry in recent years, but the associated geometrical results have been proved in an *ad hoc* manner. In this extended abstract we examine a unifying framework for visibility concerns. In particular, we use the concepts of convexity and aligned spaces to capture a generalized notion of visibility. We then investigate the relationship between kernels and skulls, in this general setting, and prove the Kernel Kernel Theorem, which implies that, under reasonable conditions, kernels are convex.

1 Introduction

An astonishing variety of "non-standard" notions of convexity in the plane have been considered in computational geometry in the past few years: *restricted orientation convexity* [9], *NESW-convexity* [8, 11], *rectangular convexity* [6, 2, 11], and *geodesic convexity* [3, 13], to name the most prominent ones. Little attention has been paid, however, to providing a general setting for geometrical results stemming from these notions. We consider convexity spaces as a candidate concept for this purpose. To this end, we define a generalized notion of visibility in convexity spaces and, based on this, we prove a general theorem relating skulls and kernels. Visibility is a well studied concept in the context of real vector spaces [1, 14, 12]. But, as the above mentioned examples illustrate, this is often too restrictive a setting.

2 The Kernel Theorem

We base the following investigation on the concept of a convexity space whose formal definition was first introduced by Levi [7]. A convexity space is intended to abstract some of the essential properties of convex sets in n dimensional Euclidian space.

Definition 2.1 *Let $\mathcal{X}$ be a set and C be a collection of subsets of $\mathcal{X}$. Then, $(\mathcal{X}, C)$ is a* convexity space *if:*

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1. $\emptyset$ and $\mathcal{X}$ are in $\mathcal{C}$; and

2. for all $\mathcal{C}' \subseteq \mathcal{C}$, we have $\bigcap \mathcal{C}' \in \mathcal{C}$, where by $\bigcap \mathcal{C}'$ we mean $\bigcap_{\mathbf{C} \in \mathcal{C}} \mathbf{C}$.[1]

$\mathcal{X}$ is called the *groundset* of the convexity space and $\mathcal{C}$ contains the "convex sets" of $\mathcal{X}$. Each set in $\mathcal{C}$ is called *$\mathcal{C}$-convex* (or convex for short if the convexity space is understood). So, the only characteristic required of convex sets is their closure under intersection. It is obvious that additional properties are needed to generalize the more intricate properties of $\mathbb{E}^n$ since a wide variety of structures satisfy the above definition.

Immediately associated with a convexity space is the convex hull operator.

Definition 2.2 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space; then, for all $\mathbf{X} \subseteq \mathcal{X}$, the* $\mathcal{C}$-hull *of $\mathbf{X}$, which is denoted by $\mathcal{C}$-hull$(\mathbf{X})$, is defined by*

$$\mathcal{C}\text{-}hull(\mathbf{X}) = \bigcap \{\mathbf{C} \in \mathcal{C} \mid \mathbf{X} \subseteq \mathbf{C}\}.$$

Definition 2.3 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space. We call $(\mathcal{X}, \mathcal{C})$ an* aligned space *if, for every nested chain $\mathcal{N} \subseteq \mathcal{C}$, the union of $\mathcal{N}$ is also convex; that is, $\bigcup \mathcal{N} \in \mathcal{C}$.*

Aligned spaces are well studied objects in the literature [4, 5, 10] but usually for quite different reasons than those stated here.

Given two distinct points p and q in the plane, their convex hull is the line segment joining them. Since this is the basis of visibility in polygons, our abstract definition of visibility is analogous.

Definition 2.4 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space and $\mathbf{X} \subseteq \mathcal{X}$. We say that two points x and y in $\mathbf{X}$* see each other *if $\mathcal{C}$-hull$(\{x, y\}) \subseteq \mathbf{X}$. We write x sees$_{\mathbf{X}}$ y in this case.*

Once having established a consistent definition of visibility it is easy to generalize the notions of starshaped sets and kernels for convexity spaces.

Definition 2.5 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space and $\mathbf{X} \subseteq \mathcal{X}$.*

1. *For $x \in \mathbf{X}$, we define $\mathcal{C}$-star$(x, \mathbf{X}) = \{y \in \mathbf{X} \mid x$ sees$_{\mathbf{X}}$ $y\}$.*

2. *$\mathbf{X}$ is* star-shaped *if $\mathbf{X} = \mathcal{C}$-star$(x, \mathbf{X})$ for some $x \in \mathbf{X}$.*

3. *$\mathcal{C}$-kernel$(\mathbf{X}) = \{x \in \mathbf{X} \mid \mathcal{C}$-star$(x, \mathbf{X}) = \mathbf{X}\}$.*

4. *$\mathbf{S} \subseteq \mathbf{X}$ is a $\mathcal{C}$-*skull *of $\mathbf{X}$ if $\mathbf{S} \in \mathcal{C}$ and there is no $\mathbf{S}' \in \mathcal{C}$ such that $\mathbf{S} \subset \mathbf{S}' \subseteq \mathbf{X}$.*

5. *$\mathcal{C}$-skulls$(\mathbf{X}) = \{\mathbf{S} \mid \mathbf{S}$ is a $\mathcal{C}$-skull of $\mathbf{X}\}$.*

Although skulls may not exist in convexity spaces, they always exist in aligned spaces. As a last visibility-related concept we need the notion of a *$\mathcal{C}$-join*.

[1] In this paper we will use $\bigcap \mathcal{F}$ to denote the intersection of all sets in a family $\mathcal{F}$ and $\bigcup \mathcal{F}$ to denote the union of all sets in $\mathcal{F}$.

Definition 2.6 *Let $(\mathcal{X},\mathcal{C})$ be a convexity space, $\mathbf{C} \subseteq \mathcal{X}$, and $x \in \mathcal{X}$, we define*

$$\mathcal{C}\text{-}join(x,\mathbf{C}) = \bigcup_{c\in\mathbf{C}} \mathcal{C}\text{-}hull(\{x,c\}).$$

The $\mathcal{C}$-*join* of a convex set $\mathbf{C}$ and a point x consists, intuitively speaking, of all the line segments between x and points c in $\mathbf{C}$. It is easy to show that the $\mathcal{C}$-*join* in the plane is *always* convex if we consider normal convexity. This is, however, not true for arbitrary convexity spaces.

Definition 2.7 *Let $(\mathcal{X},\mathcal{C})$ be a convexity space. $(\mathcal{X},\mathcal{C})$ is said to satisfy the* $\mathcal{C}$-join condition *if, for all $x \in \mathcal{X}$ and $\mathbf{C} \in \mathcal{C} \setminus \{\emptyset\}$, $\mathcal{C}$-join$(x,\mathbf{C})$ is convex.*

After this preparation we can now state and prove the Kernel Theorem. It gives a complete characterization of those convexity spaces for which the kernel of a set $\mathbf{X}$ is given by the intersection of all skulls in $\mathbf{X}$.

Theorem 2.1 (The Kernel Theorem) *Let $(\mathcal{X},\mathcal{C})$ be a convexity space. Then, we have, for all $\mathbf{X} \subseteq \mathcal{X}$,*

$$\mathcal{C}\text{-}kernel(\mathbf{X}) = \bigcap \mathcal{C}\text{-}skulls(\mathbf{X})$$

if and only if the following three conditions hold:

i. $(\mathcal{X},\mathcal{C})$ *is an aligned space.*

ii. *For all $x \in \mathcal{X}$, for all $\mathbf{C} \in \mathcal{C}$, $\mathcal{C}$-join$(x,\mathbf{C})$ is convex.*

iii. *For all x, $y \in \mathcal{X}$, $\mathcal{C}$-hull$(\{y\}) \subseteq \mathcal{C}$-hull$(\{x\}) \cup \{y\}$.*

Proof: We only proof that if Conditions (i)-(iii) hold, then $\mathcal{C}$-*kernel*$(\mathbf{X}) = \bigcap \mathcal{C}$-*skulls*$(\mathbf{X})$. For brevity, let $K = \mathcal{C}$-*kernel*$(\mathbf{X})$ and $I = \bigcap \mathcal{C}$-*skulls*$(\mathbf{X})$. We split the proof into two parts.

$K \subseteq I$. If $K = \emptyset$, this holds vacuously, so assume that $K \neq \emptyset$. Consider $p \in K$; we prove that $p \in I$. Let $\mathbf{S}$ be a skull in $\mathcal{C}$-*skulls*$(\mathbf{X})$ and s a point in $\mathbf{S}$; since $p \in \mathcal{C}$-*kernel*$(\mathbf{X})$, we have p *sees*$_{\mathbf{X}}$ s. Thus, $\mathcal{C}$-*hull*$(\{p,s\}) \subseteq \mathbf{X}$ and so $\mathcal{C}$-*join*$(p,\mathbf{S}) = \bigcup_{s\in\mathbf{S}} \mathcal{C}$-*hull*$(\{p,s\}) \subseteq \mathbf{X}$; furthermore, $\mathcal{C}$-*join*$(p,\mathbf{S})$ is convex, by assumption. But $\mathbf{S}$ is a maximal inscribed convex set of $\mathbf{X}$; therefore, $\mathcal{C}$-*join*$(p,\mathbf{S}) = \mathbf{S}$, $p \in \mathbf{S}$, and hence $p \in I$.

$I \subseteq K$. Again assume that $I \neq \emptyset$ and consider $p \in I$ and an arbitrary point $x \in \mathbf{X}$. We have to show that p *sees*$_{\mathbf{X}}$ x. Since $\mathcal{C}$-*hull*$(\{p\}) \subseteq I \subseteq \mathbf{X}$, we have $\mathcal{C}$-*hull*$(\{x\}) \subseteq \{x\} \cup \mathcal{C}$-*hull*$(\{p\}) \subseteq \mathbf{X}$ by Condition (iii). Also, since $\mathcal{C}$-*hull*$(\{x\}) \subseteq \mathbf{X}$, we know that there is an $\mathbf{S}_x \in \mathcal{C}$-*skulls*$(\mathbf{X})$ with $\mathcal{C}$-*hull*$(\{x\}) \subseteq \mathbf{S}_x$. Now $p \in I \subseteq \mathbf{S}_x$ and, thus, $\mathcal{C}$-*hull*$(\{p,x\}) \subseteq \mathbf{S}_x \subseteq \mathbf{X}$. Therefore, p *sees*$_{\mathbf{X}}$ x and $p \in K$.

□

As an immediate consequence we get the following corollary.

Corollary 2.2 *Let $(\mathcal{X},\mathcal{C})$ be a convexity space that satisfies the conditions of the Kernel Theorem; then, for all $\mathbf{X} \subseteq \mathcal{X}$, $\mathcal{C}$-kernel$(\mathbf{X})$ is convex.*

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