Geometry of Bivariate Splines

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EXTENDED ABSTRACT. A classical problem in computational geometry concerns "polyhedral pictures" or "terrains" - plane pictures of the vertices and "faces" which claim to represent piecewise linear, continous surfaces in 3-space. The questions arise at a number of different levels: Is there a non-trivial surface (not lying in a single plane)? Is there a "sharp surface", with each pair of adjacent faces in distinct planes (Figure 1)? If yes, what is the dimension of the space of such polyhedral surfaces? Are there convex srufaces (i.e. is this a generalized voranoi diagram)? For these problems, there are both combinatorial algorithms, for vertices in "generic position" and geometric algorithms for vertices in "special position" ([Sugihara], [Whiteley 1,3,4]).

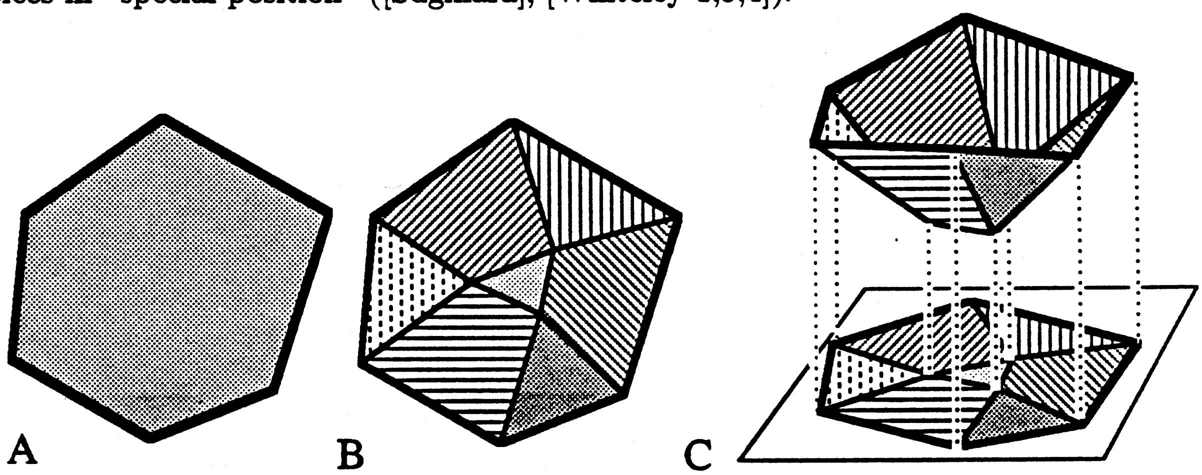

Figure 1.

This problem has a natural generalization which relates to current studies in approximation theory and in geometric design. Consider a plane picture of vertices and polygonal faces forming a cell complex Δ. We can ask: Is the projection of a piecewise quadratic, globally continuous (C^0) surface, or a piecewise quadratic surface with continuous tangent planes (C^1). In general we can ask about piecewise degree d surfaces which are globally C^r - the space $S^r_d(\Delta)$ of *bivariate* C^r_d*-splines* ([Alfeld] [Chui])

Recent research has shown that we can transfer a number of techniques from "picture recognition" to the study of bivariate C_d^r-splines [Whiteley 2,5,6].

We present some recent, unpublished results using illustrations for the particular case of $S_2^1(\Delta)$. The analysis comes in two levels: a combinatorial analysis of the plane cell complex which produces standard answers for "generic positions" (i.e. almost all values for the vertices) and a geometric analysis which studies the "special positions" where the generic values do not apply.

For "generic positions" we give the dimension of $S_2^1(\Delta)$ for any triangulated plane cell complex (Figures 2A,A'), extending results of [Billera, Whiteley 2,5]. We also describe some results for non-triangulated cell complexes (Figures 2B,B'). The techniques used combine inductive techniques in graph theory and simple matrix arguments - producing combinatorial results which have a deep analogy with statics of spatial frameworks. Because of the similarity in techniques, several standard algorithms also transfer.

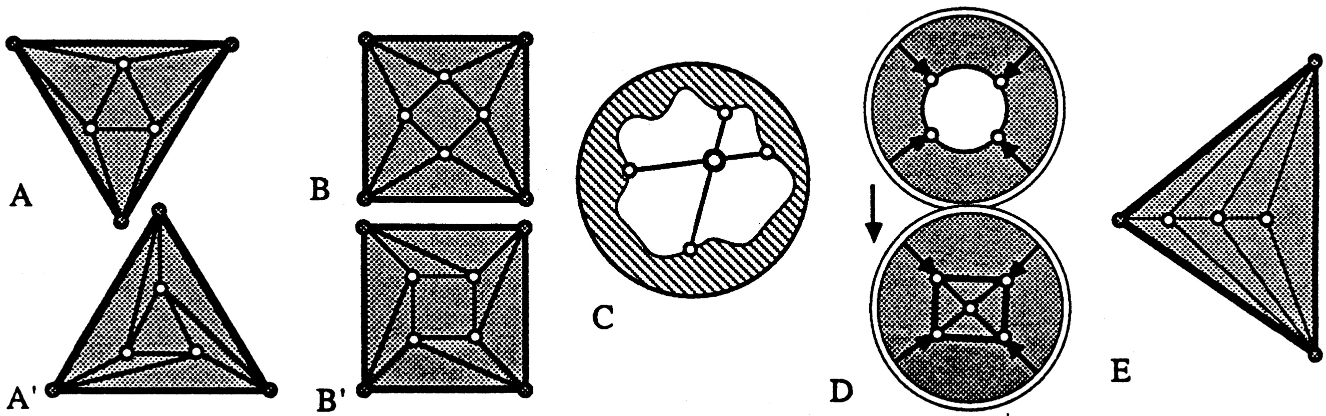

Figure 2.

For "special positions", we analyse several different types of problems:

(i) Special positions of critical configurations (Figures 2A,B not A′,B′);

(ii) Singular vertices and their effect on the dimensions (Figure 2C;

(iii) Extrapolation from a cell complex into an unfilled hole (Figure 2D);

(iv) Interpolation of a surface to fixed heights over the vertices (Figure 2E).

Previous work has emphasized algebraic results which apply to $S_d^1(\Delta)$ for $d \geq 4$ (more generally $S_d^r(\Delta)$, $d \geq 3r+1$). Our more geometric results apply well to smaller values of d, specifically $r+1 \geq d \leq 2r$. Nevertheless the results also have important consequences for the use of splines for all d over domains in 3-space.

These results raise some conjectures for combinatorial algorithms and geometric algorithms to detect the presence of 'sharp' splines, and non-trivial splines, extending the results for polyhedral scenes.

Although the problems originated in approximation theory, our study corresponds much more closely to the problems of computer aided design and creation of specific geometric objects of the desired smoothness.

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