Computing Polygonal Chords and the Farthest Visibility Polygons

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1 Introduction

Given a polygon Q and a point p in the plane, we wish to find both the longest chord and the shortest chord of Q that passes through or extends through the point p. Throughout this paper, it is assumed that Q is an n-sided simple polygon, while the point p may be interior to, on the boundary of, or exterior to Q.

1.1 Motivation

The computation of polygonal chords with respect to a point is related to the computation of diameters and diagonals. The diameter of a point set S [11] is the longest line segment connecting two points in S. It can be computed in $O(n \log n)$ time. A ***proper chord*** of a polygon Q is a line segment that lies entirely in Q. The diagonal of a polygon is a longest proper chord that connects two vertices. An $O(n \log^3 n)$ time and $O(n)$ space algorithm for computing diagonals is discussed in [1]. The problem of finding the longest proper chord of a polygon can be solved in $O(n^{1.98})$ time [3].

The diameter, diagonal, and convex hull of a point S are examples of so-called ***view-independent*** properties of S. On the other hand, the visibility polygon of Q with respect to a view-point p is an example of so-called ***view-dependent*** property of the polygon Q with respect to the point p. The longest chord through p of Q can be considered to be a view-dependent version of a diagonal.

In this paper we also introduce the concept of the ***farthest visibility polygon***, which is a view-dependent property of a polygon. The farthest visibility polygon (with respect to the view-point p) can be considered as a dual of the (nearest) visibility polygon [6][9].

1.2 Definitions and Overview

When p is an internal point or a boundary point of Q, we define a ***polygonal chord*** (or simply ***chord***) through p to be a line segment that goes between two boundary points of Q and contains the point p. But if p is external to Q, it is not sensible for a polygonal chord to pass through the point p. Therefore, we enlarge the notion of polygonal chords through p by admitting all the chords that lie on a line that passes through the possibly external point p. Such chords are said to *extend through* p.

A ***chord*** of the polygon Q connects any two points on the boundary of Q. A ***proper chord*** of Q connects two points that are on the boundary of Q, such that the chord segment has no point external to Q. There may be multiple proper chords contained in a chord.

To begin, suppose that p is an interior point of an n-sided polygon Q; In section 2, we prove that if Q is convex, then both a longest chord through p and a shortest chord through p can be found in $O(n)$ time. This involves showing that the function that describes the length of a chord through a point p and between a pair of edges of Q is *unimodal*, and has a unique global minimum. Therefore, a longest chord of Q passing through the point p can be found by examining all the n chords that go from a vertex of Q and pass through the point p. Similarly, a shortest chord that passes through the point p can be found by examining up to n pairs of edge segments of Q.

We can treat non-convex polygons by extending the results in section 2. First of all, Theorem 1 also holds when Q is a star-shaped polygon, and p is in its visibility kernel, or simply ***kernel***, which is the set of points in Q that can see every boundary point of Q. And even if Q is not star-shaped or p is not in the kernel of Q, we still can find a longest chord and a shortest chord that pass through the interior point p in $O(n)$ time. This involves constructing the (nearest) ***visibility polygon*** of Q.

What if p is not interior to Q? If p is on the boundary of Q, we can prove that the same methods apply for

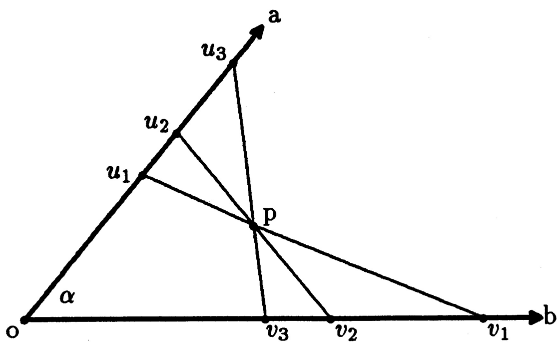

Figure 1: three polygonal chords

finding both longest or shortest proper chords passing through p.

If p is external to Q, then all the associated shortest chords that extend through p must have length 0. This is because we can always find a chord extending through the external point p that contains a one-point proper chord, which has to be a vertex, of the polygon Q.

We can find a longest chord that extends through either an external or internal point p in $O(n)$ time, by first computing the so-called *farthest visibility polygon* of Q with respect to the point p. This is discussed in section 3. Note that we may need $O(n \log n)$ time to generate all the pairs of edges that admit one or more proper chords that extend through an external point p.

In section 4, the computation of shortest chords through an internal point p is discussed. In section 5, a generalization of polygonal chords, called *broken chords* is introduced. A broken chord through p consists of a pair of line segments with a common end-point p.

2 Longest chords that pass through a non-external point

2.1 The internal point theorem

In Figure 1, we are given two semi-infinite rays $\vec{oa}$ and $\vec{ob}$ from a point o, and an internal point p lying between these two rays, where angle $\angle(aob) = \alpha \in (0, \pi)$.

Let u_1, u_2, and u_3 be any three distinct points on $\vec{oa}$, with u_2 between u_1 and u_3. Let v_1, v_2, and v_3 be three distinct points on $\vec{ob}$, such that the threes line that go from the point u_i to the point p intersect $\vec{ob}$ at v_i, for $i = 1, 2, 3$. Let $|\overline{uv}|$ denote the length of the line segment $\overline{uv}$. We wish to prove the following.

Theorem 1: $|\overline{u_2v_2}| < \max\left(|\overline{u_1v_1}|, |\overline{u_3v_3}|\right)$.

Proof sketch: The key to this proof is to express the chord-length function $y(\theta) = |\overline{qr}|$ in terms of an angular parameter θ, and to show that the function $y(\theta)$ is unimodal by showing its second derivatives are always positive.

2.2 Computing longest chords through a non-external point

Let us first consider computing a longest chord of a convex polygon Q passing through an internal or boundary point p. Such a chord is always proper. By Theorem 1, a longest chord of a convex polygon Q through an internal point p must also go through a vertex of Q. For a proper chord through p to go through a vertex of Q, it is sufficient that Q is star-shaped and that p is in its *kernel*. It is possible to find the lengths of all the proper chords passing through p and a vertex of Q in $O(n)$ time by using a technique similar to that used in finding all $O(n)$ *antipodal pairs* [11] of a convex polygon.

The visibility region of a polygon Q with respect to an internal point p is the portion of Q that is *visible* from the point p. This corresponds to a polygon, denoted by $NV(Q,p)$, called the (nearest) *visibility polygon* of Q with respect to an internal point p. It can be built in $O(n)$ time [9]. The visibility polygon $NV(Q,p)$ is a bounded star-shaped polygon with at most n vertices and the point p lies in its kernel.

Theorem 1 is also true even if p is a *boundary* point of Q. Therefore, a longest proper chord of a polygon Q can be found in $O(n)$ time for p internal to or on the boundary of Q.

3 Longest chords that extend through an external point

Now, let us consider longest chords of a polygon Q that are constrained to lie on a line passing through an external point p. Theorem 1 no longer holds if the point p is *external* to Q. This is because the chord-length function $y(\theta)$ of the proper chords between any two infinite rays, constrained to pass through a point p which is outside the wedge formed by the rays, is either *bimodal* or monotonically increasing. Theorem 2 below states that the chord-length function either has exactly one local maximum and one local minimum, or is a monotonically increasing function for some parameter θ.

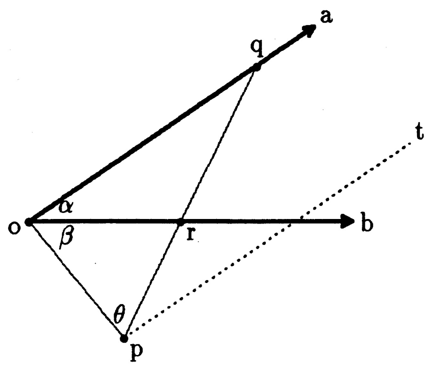

Figure 2: chord from an external point

3.1 The external point theorem

In Figure 2, we are given two semi-infinite rays $\vec{oa}$ and $\vec{ob}$ extending from a point o, and suppose we have an *external* point p lying outside the cone formed by these two rays and their inverse rays.

Consider the chord $\overline{qr}$ from a point q on $\vec{oa}$ to a point r on $\vec{ob}$ that extends through the external point p. Let $\theta = \angle(opr)$, and let $y(\theta) = |\overline{qr}|$. Thus, $y(\theta)$ is the length of the chord $\overline{qr}$. Note that $\theta \in [0, \pi - \beta - \alpha]$, where $\alpha = \angle(aob)$, and $\beta = \angle(bop)$. The two rays $\vec{oa}$ and $\vec{pt}$ are parallel, and the angle $\angle(opt) = \pi - \beta - \alpha$. Also note that $0 < \alpha < \pi$, and $0 < \beta < \beta + \alpha < \pi$. When q and r are chosen so that $\theta = 0$, then the length of the chord $\overline{qr}$ between the two rays $\vec{oa}$ and $\vec{ob}$ such that $\overline{qr}$ extends through the point p is globally minimal, with the chord-length $y(0) = 0$. When $\theta = \pi - \beta - \alpha$, q and r are chosen so that the chord-length $y(\theta)$ becomes infinite. We wish to show the following.

Theorem 2: $y(\theta)$ is either a monotonically increasing function or a bimodal function of the parameter θ, in the interval $0 \le \theta \le \pi - \beta - \alpha$.

Proof sketch: Although this theorem is similar to Theorem 1, the proof is somewhat more complicated. We can show that the chord-length function $y(\theta)$ is of the form $\sin(\theta)/(\sin(\theta + \beta + \alpha) * \sin(\theta + \alpha))$. We may then observe that the function y is either unimodal or bimodal, by means of a series of reductions to simpler forms.

3.2 Longest proper chords that extend through an external point

By Theorem 2, a longest *proper chord* of the polygon Q which extends through an *external* point p need not pass through a vertex of Q. By resorting to standard numerical technique, we still can find the extreme values of the function $y(\theta)$.

If the polygon Q is convex, then there are at most n pairs of edge-fragments that need to be considered, and we can find the longest proper chord that extends through the point p in $O(n)$ time. Note that every vertex i in Q can admit at most two proper chords that extend through the internal or external point p and run between an edge of Q and the vertex i.

Unfortunately, we know of no algorithm to generate all the $O(n)$ pairs of edges in linear time. By arranging the edges of the polygon Q into a so-called *angular segment tree* with respect to the center p, we can generate all the $O(n)$ pairs of edges that admit at least one proper chord that extends through p in $O(n \log n)$ time. An angular segment tree with center p of a polygon Q stores all the pairs of end-points of edge segments of Q, such that the end-points are angularly sorted with respect to p. Such a segment tree can be constructed in $O(n \log n)$ time.

Note that if the point p is non-external to Q, we still can compute its proper chords that extend through the point p in the same manner. This is because the so-called *invisible portion* of the polygon Q as seen from p, defined to be the point set $Q - NV(Q,p)$, treats the point p as an external point. Thus we have the following theorem.

Theorem 3: A longest *proper chord* of a n-sided simple polygon Q that extends through *any* point p can be found in $O(n \log n)$ time. A longest *proper chord* that actually passes through a given internal point or boundary point of Q can be found in $O(n)$ time.

3.3 Farthest visibility polygons

The farthest visibility polygon of a polygon Q with respect to a point p, denoted by $FV(Q,p)$, is defined as follows:

1. Construct a circle C centered at p which surrounds Q.
2. Imagine light shining *from* the circumference of C toward the point p. The edge fragments of Q which are illuminated can be connected end-to-end (according to the polar angles of the end-points around the center p) to form the farthest visibility polygon $FV(Q,p)$ with respect to the point p.

$FV(Q,p)$ is formed by edge-fragments of Q that are *not* obscured by any other edge of Q, as the light radiates from C toward the point p.

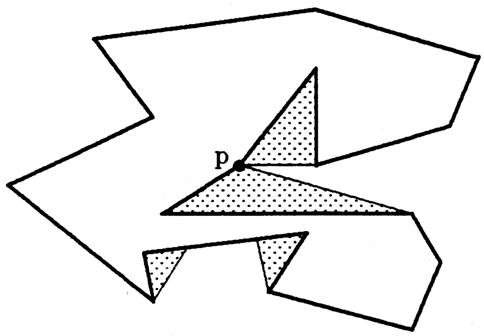

Figure 3: a farthest visibility polygon

Let us first assume that p is a point inside $conv(Q)$, the convex hull of Q. In this case, $FV(Q,p)$ will be a bounded star-shaped polygon, with at most n edges and p being a point in its kernel. For example, in Figure 3, the polygon Q has bold solid edges, and the polygon $FV(Q,p)$ is the shaded area plus the entire area of Q.

The polygon $FV(Q,p)$ can be found in $O(n)$ time. The algorithm is similar to the algorithm for the construction of the (nearest) visibility polygon [9][7].

Given $FV(Q,p)$ with p inside $conv(Q)$, we can compute a longest proper chord through p. This longest proper chord will be a longest chord, proper or not, of Q that extends through p. If p is outside $conv(Q)$, we still can find $FV(Q,p)$ in $O(n)$ time, and compute its longest chord in linear time.

3.4 Some properties of visibility polygons

Let edge $e = \overline{ab}$, $conv(Q)$ be the convex hull of Q, $NV(Q,p)$ and $FV(Q,p)$ are the nearest and farthest visibility polygons respectively, with respect to an internal point p.

1. $\bigcap_{p\in e} NV(Q,p) = NV(Q,a) \cap NV(Q,b) = NV(NV(Q,a),b)$
2. $\bigcup_{p\in e} NV(Q,p) \supseteq NV(Q,a) \cup NV(Q,b)$
3. $\bigcup_{p\in e} FV(Q,p) = FV(Q,a) \cup FV(Q,b) = FV(FV(Q,a),b)$
4. $\bigcap_{p\in e} FV(Q,p) \subseteq FV(Q,a) \cap FV(Q,b)$
5. $NV(Q,p) \subseteq Q \subseteq FV(Q,p) \subseteq conv(Q)$

Note that both items 1 and 3 can be computed in $O(n)$ time [9]. Item 2, the so-called ***weakly*** (nearest) visibility polygon, can be computed in $O(n \log n)$ time [10][4] or $O(n)$ time plus the time to triangulate Q [8]. Item 4, the so-called ***strongly*** farthest visibility polygon, can be computed in $O(n \log n)$ time using duality, as well as Chazelle and Guibas's algorithm [4] for item 2 above.

4 Shortest chords through a non-external point

Given any polygon Q, the shortest chords through an *external* point p of Q must have length 0. Therefore, we shall consider cases where the point p is internal to or on the boundary of Q.

Given a star-shaped polygon Q and a point p in its kernel, the edge segments $\overline{a_1a_2}$ and $\overline{b_1b_2}$ are said to be an *antipodal pair of segments* of Q through p if and only if:

1. a_1, p, and b_1 are collinear, and a_2, p, and b_2 are collinear.
2. $\overline{a_1a_2}$ is a subset of an edge of Q, and $\overline{b_1b_2}$ is a subset of another edge of Q.
3. at least one of the points a_1 and b_1 is a vertex of Q, and at least one of the points a_2 and b_2 is a vertex of Q.

For a star-shaped polygon Q, with the point p in its kernel, there are at most n antipodal pairs of segments. A shortest proper chord through p between an antipodal pair of edge-segments of Q, can be found in time independent of n. This can be done by applying Newton's method to find a root of the chord-length function $y(\theta)$ described in Theorem 1. Therefore, by scanning the n antipodal pairs of segments in order, a shortest chord of Q through the point p can be found in $O(n)$ time.

What if the polygon Q is not a star-shaped polygon with p in its kernel? Construct the visibility polygon $NV(Q,p)$ of the polygon Q with respect to the point p in $O(n)$ time. Then the shortest chord of Q through the point p can be computed as the shortest chord of $NV(Q,p)$ through p.

5 Broken chords: a generalization

A polygonal chord is a line segment that connects two boundary points of a polygon. We wish to generalize the notion of chords by introducing the notion of a broken chord as follows:

A ***broken chord*** with a wedge-angle γ broken at an internal point p of a polygon Q is made up of a pair of line segments $\overline{sp}$ and $\overline{pt}$, where s and t lie on the boundary of Q, and the angle $\angle(s,p,t) = \gamma \in [0,\pi]$. It is written as $\widehat{spt}$. A broken chord of Q with a wedge angle γ broken at p is called a (γ,p) broken chord of Q.

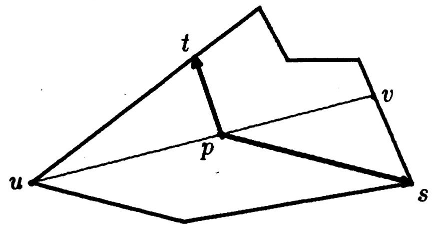

Figure 4: a broken chord

For example, in Figure 4, $\overline{uv}$ is the longest chord of the polygon Q, which passes through the point p and a vertex u. For $\gamma = \frac{2}{3}\pi$, the longest (γ, p) broken chord is formed by the pair of segments $\overline{sp}$ and $\overline{pt}$, where the point s is a vertex of Q.

Consider the (γ, p) broken chords broken at the point p which have a fixed wedge-angle γ, where $0 \leq \gamma \leq \pi$. When $\gamma = \pi$, a (γ, p) broken chord is the same as a polygonal chord that passes through p. When $\gamma = 0$, a (γ, p) broken chord coincides with a line segment connecting p to a boundary point, but the broken chord has twice the length. It turns out that the chord-length function for describing a broken chord $\widehat{spt}$ between a pair of edges (or a single edge) is *unimodal*. The proof is almost identical to that of Theorem 1.

Let us assume that the polygon Q is star-shaped, and the point p is in the kernel of Q. All (γ, p) broken chords of Q (through the point p), for all angles $\gamma \in [0, \pi]$, must lie entirely in Q. With results from similar to those of sections 2-4, we are able to compute longest or shortest (γ, p) broken chords of Q in $O(n)$ time, where the edge-angle γ is fixed.

What if the polygon Q is not star-shaped or p is not in its kernel? When we wish to compute longest or shortest (γ, p) broken chords that are proper, all we need is to compute the visibility polygon $NV(Q, p)$ of the input polygon Q with respect to the point p. Then a longest and shortest (γ, p) broken chord of $NV(Q, p)$ can be computed. When we wish to compute longest or shortest (γ, p) broken chords that are not necessarily proper, all we need is to compute the farthest visibility polygon $FV(Q, p)$ of the input polygon Q with respect to the point p. Therefore we conclude that both the longest and the shortest (γ, p) broken chords of $FV(Q, p)$ can be computed in $O(n)$ time.

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6 References

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