LINK LENGTH OF RECTILINEAR WATCHMAN TOURS IN GRIDS
(Extended Abstract)

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1. Introduction

The link length of a polygonal path connecting two points in a polygon is defined to be the number of straight line segments constituting the path. The link distance of two points is the minimum link length of any polygonal path connecting the points. Alternatively, it is the number of turns that a mobile unit will need to take when traversing a minimum-turn-walk connecting the two points in order to move from one point to the other. This distance was introduced by [Suri, 1986] and was designed to measure the cost of moving along a path in a simple polygon when straightline motion is easy but turns are expensive. Subsequent studies of this metric concentrated on algorithms for constructing the link center of an n vertex polygon (in time $O(n^2)$), i.e. the set of points x inside the polygon whose maximal link distance to any other point inside the polygon is minimized [Lenhart et al, 1987], finding a point in the link center of the polygon (in time $O(n\log n)$) [Djidjev et al, 1989], or finding the link diameter of the polygon, i.e. the maximal link distance between any two points (in time $O(n\log n)$) [Suri, 1987].

Klee [O'Rourke, 1987] first proposed the Gallery Watchman Problem in which we are asked to determine the minimum number of stationary watchman required so that every point in a polygonal gallery containing polygonal holes is seen by at least one watchman at any time. A number of researchers have considered restricted versions of this problem (see [O'Rourke, 1987]). Recently, [Hoffman, 1990] showed that $[n/4]$ watchman are necessary and sufficient for the case of rectilinear polygons with an arbitrary number of holes. [Ntafos, 1986] studied the question of gallery watchman in incomplete grids. He gave a polynomial time algorithm for placing watchmen in incomplete 2-dimensional grids and showed the problem is NP-complete in the case of incomplete 3-dimensional grids. A related problem was introduced by [Chin and Ntafos, 1986], that of optimum watchman routes. Here there is a single mobile watchman and we are asked to determine a minimum length route for the watchman with the property that every point in the gallery is visible from at least one point along the route.

In this paper, we introduce a further variation on the above problems which we call the Minimum Link Length Watchman Tour: Given a polygonal gallery and a series of distinguished points in the gallery determine the link length of a minimum link length path for a watchman which visits all the points. It is known [Clote, 1989] that determining the minimum link length

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watchman tour in a general polygonal gallery is NP-complete, since the "edge embedding on a grid problem" [Gavril, 1977] can be reduced to it. The discussion below is restricted to rectilinear tours which visit all the vertices of a complete d-dimensional grid-like gallery. This restriction greatly simplifies the problems considered above but as will be seen the problem at hand remains interesting even in this case. We give the exact solution for the problem for 2-dimesional grids and give nontrivial bounds for d-dimensional grids, for $d>2$. Despite the seeming simplicity of the question, the exact solution for 3-dimesional grids remains open.

2. Preliminaries

Next we introduce some definitions and notations. The complete d-dimensional grid of size n, denoted G_n^d, is the graph with vertices $v := (v_1, v_2, ..., v_d)$ such that $1 \leq v_i \leq n$, for $i = 1,...,d$, and edges (u, v) such that $\sum_{i=1}^{d} |u_i - v_i| = 1$. To every rectilinear walk P, we associate the unique partition (called the **rectilinear partition**) of P, $L_1, L_2, ..., L_s$, consisting of the "maximal straight line segments" of P. The number $s := s(P)$ is the **link length** of the walk P. It is not hard to see that $s(P)$ is exactly the number of times one must change direction moving along P in order to traverse all the vertices of P. A rectilinear walk traversing all the vertices of the grid G is called a **complete**, rectilinear tour. We also define the rectilinear number of the grid by

$$s(G) = \min\{s(P) : P \text{ is a rectilinear walk traversing all vertices of } G\}.$$

The present paper studies the problem of determining $s(G)$ for the complete multidimensional grids. In particular, we estimate the value of the quantity $s(G_n^d)$, for $d \geq 2$, $n \geq 1$. A straightforward estimate is

$$\frac{n^d - 1}{n - 1} \leq s(G_n^d) \leq n \cdot s(G_n^{d-1}) + n - 1. \tag{1}$$

The main result of the paper is to show that the actual value of $s(G_n^d)$ satisfies much stronger upper and lower bounds than those implied above.

3. Main Results

We begin with the simple case $d = 2$. We prove the following theorem.

Theorem 3.1. For all $n \geq 2$, $s(G_n^2) = 2n - 1$.

Proof. To prove $s(G_n^2) \leq 2n - 1$ consider the straightforward "snake-like" tour of the grid, across the first row from left to right, down one, across the second row from right to left, etc. It remains to prove that $s(G_n^2) \geq 2n - 1$. Put $s = s(G_n^2)$, let P be a rectilinear tour of G_n^2, with $s = s(P)$ and let $L_1, L_2, ..., L_s$ be the rectilinear partition of P. Let h (respectively, v) be the number of horizontal (respectively, vertical) L_i's. Clearly, $s = h + v$. By definition of rectilinear partitions, for all $i < s$, if L_i is horizontal (respectively, vertical) then L_{i+1} is vertical (respectively, horizontal). Consequently, $|h - v| \leq 1$. Assume that $h \leq n - 1$. This means that there is a horizontal line, say L, of the grid G_n^2 which is not traversed by any of the horizontal L_i's. Consequently, the n vertices of L must be traversed by n vertical L_i's. This implies that $v \geq n$. It follows that $h = n - 1$ and $v = n$. A symmetric reasoning shows that if $v \leq n - 1$ then $v = n - 1$ and $h = n$. In either case we conclude $s \geq 2n - 1$. •

It is easy to see that the same argument will work for the $m \times n$-grid. We single out this simple observation as a corollary since it will be useful in the sequel.

Corollary. Exactly $2\cdot\min(m,n)-1$ turns are necessary and sufficient in order to solve the rectilinear tour problem for the $m\times n$ grid. •

Next we consider the case $d=3$. The observation (1) made above implies $n^2+n+1 \le s(G_n^2) \le 2\cdot n^2-1$. The following theorem improves both the lower and upper bounds.

Theorem 3.2. There is a constant $c>1$, such that for all $n\ge 3$,

$$c\cdot n^2 \le s_1(G_n^3) \le \frac{3}{2}\cdot(n^2+n)-1$$

Sketch of proof of upper bound. The tour of the three dimensional grid can be described in the following way. Traverse the bottom horizontal plane grid by moving on its periphery from the outside to the inside and covering each time all of the corresponding vertices. Proceed this way until you cover vertices of the plane grid up to a depth of $[n/4]$ vertices. This leaves an $\lceil n/2\rceil\times\lceil n/2\rceil$ square-grid in the middle whose vertices must be covered. At this point finish with this plane, draw a vertical line (in order to get connected with the next horizontal plane) and start moving along this new horizontal plane grid, covering its vertices in a similar way, except that now you move from the inside to the outside. When you finish traversing its outermost vertices, draw a vertical line and move to the next plane grid, and so on. Proceed this way until you cover the top horizontal plane with similar straight lines. At the end of traversing the top plane grid you are left with a parallelepiped grid of dimensions $\lceil n/2\rceil\times\lceil n/2\rceil\times n$ standing in the middle of the three-dimensional grid G_n^3 and whose vertices must be traversed. To traverse the parallelepiped we think of it as consisting of $\lceil n/2\rceil$ $\lceil n/2\rceil\times n$ plane grids each parallel to the yz-plane. Using the algorithmic construction in the corollary to theorem 3.1 we can see that we need exactly $n-1$ straight lines to traverse each of these planes. Counting the number of lines used in this construction proves the upper bound.

Sketch of the proof of the lower bound. The idea behind the straightforward lower bound in (1) is that a single rectilinear segment can a visit at most n vertices and there are a total of n^3 vertices. To improve upon this we show that not all of the line segments can visit n vertices. The crucial observation we use here is that for any long line segment passing through a central part of the grid there must correspond a short line segment when the line turns onto a plane perpendicular to itself.

Consider a sub-grid with side $n-2k$ centered within G_n^3. For any line segment passing through a point in this grid, the next line segment in the tour must be of length less than or equal to $n-k$. Using this fact we are able to show that

$$s(G_n^3) \ge \frac{n^3-k(k+1)(3n-2k-1)+6n-7}{n-k/2}.$$

Setting $k=\alpha\cdot n$, simplifying, and maximizing the resulting fraction (with respect to α) we obtain after some calculations that $s(G_n^3)\ge(1.023)\cdot n^2$, which proves the existence of a constant $c>1$ satisfying the lower-bound. •

The results above may be generalized to give nontrivial bounds for d-dimensional grids where $d\ge 3$.

Theorem 3.3. For all $0<\varepsilon<1$, for large enough n, the following inequalities hold asymptotically in d,

$$1+\frac{1}{2}\cdot[1-\exp[-1/d(d-1)]] \le \frac{s(G_n^d)}{n^{d-1}} \le 1+\frac{1}{2}\cdot\frac{1}{(d-3)^{1-\varepsilon}}+\exp[-(d-3)^{\varepsilon}].\ \bullet$$

4. Conclusions

We have studied the asymptotic behavior of the link length of rectilinear walks traversing all the vertices of multidimensional grids and have given nontrivial bounds of the optimal link length of such rectilinear walks. Exact bounds for grids of dimension 3 or greater are still unknown. The results for the 2 and 3 dimensional cases discussed above (which are, of course, often notoriously misleading) lead us to conjecture

$$s(G_n^d) = \frac{d}{d-1}\, n^{d-1} + O(n^{d-2}).$$

Questions concerning the link length of tours in incomplete grids or more general polygons are entirely open.

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