A Reduced Grid for Rectilinear Steiner Minimal Trees

Extended Abstract

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Abstract

Hanan has shown that a rectilinear Steiner minimal tree can always be constructed using only edges on the grid formed by horizontal and vertical lines through each of the sites to be connected. This grid, with $O(n^2)$ edges and vertices, is used in several optimal algorithms for rectilinear Steiner trees. We show that there is a reduced grid with expected $O(n)$ edges and vertices. This is accomplished using a theoretical method developed by Hwang for decomposing rectilinear Steiner minimal trees.

Further, it is shown that a vertical or horizontal line crosses the reduced grid at an expected $O(\sqrt{n})$ points. This fact can be exploited in the optimal rectilinear Steiner tree algorithm of Thompson, Deneen and Shute [6] to reduce the expected run time from $n^{O(\sqrt{n})}$ to $2^{O(\sqrt{n})}$.

1 Introduction

Many practical problems require interconnecting a set of sites while holding the total length of the interconnections to a minimum. Examples arise in designing telephone and power distribution networks and laying out the interconnections between various components of an integrated circuit. If the problem allows junctions at points other than the original sites, such points are called *Steiner points*. The general problem of finding a minimal length network that interconnects all of the sites is called the *Steiner problem*, and the minimal length network is called a *minimal Steiner tree* for the set of sites. For practical reasons, the interconnections in an integrated circuit layout are usually constrained to consist of horizontal and vertical line segments in a plane. With this constraint the problem is known as the *rectilinear Steiner problem* and the minimal length network is called a *rectilinear Steiner minimal tree* for the set of sites. Garey and Johnson [3] have shown that the rectilinear Steiner problem is NP-complete.

Hanan [4] has shown that a rectilinear Steiner minimal tree can always be constructed using only edges on the grid formed by horizontal and vertical lines through each of the sites. This grid, with $O(n^2)$ edges and vertices, is used in Yang and Wing's [8] algorithm for rectilinear Steiner minimal trees. We show that there is a reduced grid with expected $O(n)$ edges and vertices that contains a rectilinear Steiner minimal tree. With the reduced grid, the run time of the of Yang and Wing's algorithm is reduced from $2^{O(n^{\frac{3}{2}})}$ to $2^{O(n)}$.

In Section 2 we present the notation and fundamental terminology that is be used in the rest of the paper. In Section 3 we describe the decomposition used by Hwang in terms more suitable for this work. In Section 4, we show that there is a reduced grid for a set of n sites with expected $O(n)$ edges and vertices. Finally, in Section 5, we describe an algorithm for computing this reduced grid in $O(n \log n)$ time.

2 Notation and terminology

Let P be a set of n points in the plane. We refer to the points of P as *sites*. A tree on P is a set of points $T \supseteq P$ in the plane such that there is a unique path in T between any pair of points of T. We restrict our attention to *rectilinear trees*, which are constructed entirely with vertical and horizontal line segments. A *vertex* of T is a point $p \in T$ that has degree three or more. A *Steiner point* of T is a vertex of T that is not a site. A *line* of T is a straight line segment contained in T. An *edge* is a line that contains no vertices except possibly at its endpoints. A *corner* is formed by two edges that intersect at right angles at a common endpoint that is not a vertex.

A rectilinear Steiner minimal tree may be modified by two kinds of operations that leave its length invariant. If T has an edge e that terminates on two parallel lines l_1 and l_2, then we can *shift* e along l_1 and l_2 to a position parallel to e. If two edges e_1 and e_2 meet to form a corner we can *flip* the corner, replacing e_1 and e_2 by parallel edges that, together with e_1 and e_2, form a rectangle. If a tree T' can be obtained from T by a sequence of shifting and flipping operations we say that T' is *equivalent* to T.

3 Decomposing a rectilinear Steiner minimal tree

Hwang [5] has described a decomposition of rectilinear Steiner minimal trees that is a crucial theoretical foundation for this paper. For a rectilinear Steiner minimal tree T on a set of sites P, a *component* of T is a maximal subset $C \subseteq T$ such that there are no sites in the interior of C. We can obtain the components of T as the closures of the connected components of the forest $T \setminus P$. See Figure 1, where the two components of a rectilinear Steiner minimal tree are emphasized with heavy lines.

As shown in Figure 2, it may happen that a shift operation on a component C results in splitting C in two. If this is possible, we say that C is *reducible*. Otherwise we say that C is *irreducible*. If each component of a tree is irreducible then we say the tree is *canonical*. If a component is reducible then each of the subcomponents contains fewer sites than the original. Thus splitting of components terminates after finitely many steps, proving the following:

Lemma 3.1 *Every rectilinear Steiner minimal tree is equivalent to a canonical tree.*

The characterization of irreducible rectilinear Steiner minimal trees is a central part of Hwang's work [5]. There it is shown that each such tree is equivalent to one in which all but possibly one of the Steiner points lie on a straight line that contains one of the sites, with sites connected on alternating sides to the Steiner points. The one exceptional Steiner point joins to one end of the line of Steiner points through a corner.

We refer to irreducible rectilinear Steiner minimal trees as *boughs*. An example of a bough whose Steiner points lie on a straight line is shown in Figure 3. Figure 4 shows a bough with an exceptional Steiner point near the left end. If no two sites lie on the same vertical or horizontal line then all boughs differ from those in Figure 3 or Figure 4 only in orientation or number of Steiner points. With degeneracies we may also have bough with four sites and a single Steiner point of degree four, as in Figure 5.

The gaps in Figures 3–5 show how to partition a bough into lines, each of which has a site as one endpoint. These lines fall into two classes. Those with interior Steiner points are called *limbs* and those with no interior Steiner points are called *twigs*.

In view of Hwang's work and the above terminology, we may restate Lemma 3.1 as follows:

Lemma 3.2 *Every rectilinear Steiner minimal tree is equivalent to a canonical tree, which can be decomposed into limbs and twigs, each terminating at a site. In a canonical tree the line attached at each Steiner point on a limb is a twig. The twigs on a limb alternate sides.*

4 Probabilistic analysis

In this section we show that limbs and twigs of a canonical Steiner tree are probably not very long. For our probabilistic arguments it is reasonable to disregard the possibility of degeneracies. Thus we assume that no two sites lie on the same horizontal or vertical line.

Now consider Figure 6, where the heavy line L is a potential limb or twig in a canonical Steiner tree T. We assume that the site on L is either above y_3 or below y_0. In Figure 6, regions B, C, and D are 45° right triangles and B extends to a larger 45° right triangle with a vertex on L and hypotenuse on the line $y = y_3$. The following lemma describes a situation that obstructs L from being a part of T.

Lemma 4.1 *In Figure 6, if there are no sites in the shaded region A but there is at least one site in each of regions B, C, and D, then the line segment on L from y_0 to y_3 cannot be entirely contained in any canonical Steiner tree T.*

Proof: Omitted.

Our method for producing a reduced grid is to form a grid from horizontal and vertical lines terminating at each site, as in Hanan's grid. But instead of extending these lines to the boundaries of a rectangle that encloses all of the sites, we determine a bound on the length of each line by searching for configurations of sites that satisfy the hypotheses of Lemma 4.1.

As in [1], our probabilistic analysis is done in the context of a Poisson process with intensity 1 on a $\sqrt{n} \times \sqrt{n}$ square. We fix dimensions in Figure 6 by setting $y_3 - y_0 = 1$ and choosing y_1 and y_2 to divide the line segment on L between y_0 and y_3 in thirds. Applying Poisson probability formulas to the various regions in Figure 6, we find there is a constant $c > 0$ such that the probability that the segment of L between y_0 and y_3 is not contained in any canonical Steiner tree is at least c. Note that c is an absolute constant, independent of n. We can put $\lfloor l \rfloor$ copies of Figure 6 along a line L of length l, without overlapping any of the regions. Thus the probability that L is contained in some canonical Steiner tree is at most $(1-c)^{\lfloor l \rfloor}$. Our main theorem can be deduced from this fact.

Theorem 4.2 *For a set of n sites in a square there is a reduced grid, consisting only of lines through the sites, which contains any canonical rectilinear Steiner minimal tree for the set of sites. For points that are generated by a Poisson process in a square, the expected number of vertices in the reduced grid is $O(n)$ and the expected number of intersections of a horizontal or vertical line with the grid is $O(\sqrt{n})$.*

Proof: Omitted.

5 An algorithm for finding a reduced grid

The regions involved in the hypotheses of Theorem 4.1 are or can be decomposed into a few rectangular or triangular regions. Thus, using range search methods described in [2, 7], we can make a single test for the hypotheses in time $O(\log n)$. From each site we can proceed in each of the four coordinate directions, testing the hypotheses, and advancing if the hypotheses are not satisfied. The results in the preceding section show that the hypotheses will be satisfied in an expected constant number of tries. Then we can terminate the line in that direction from that site. With n sites, the expected run time for the algorithm is then $O(n \log n)$.

References

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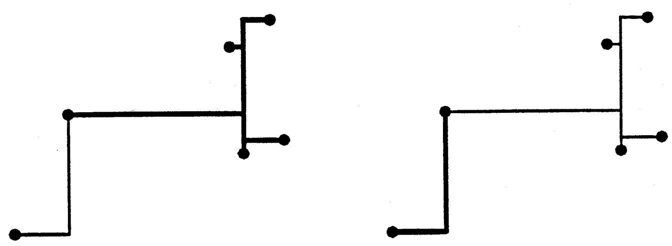

Figure 1: Components of a rectilinear Steiner minimal tree.

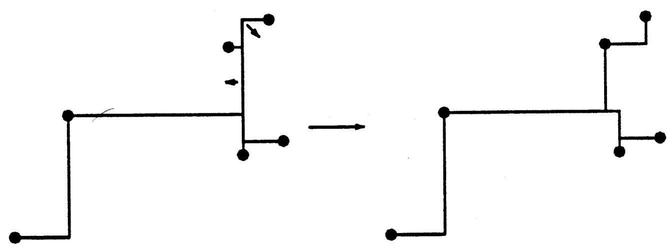

Figure 2: Reduction of a component.

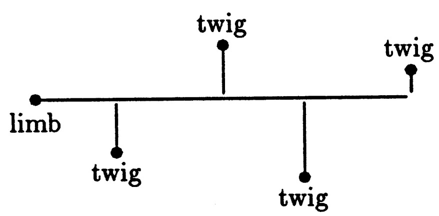

Figure 3: A bough and some of its parts.

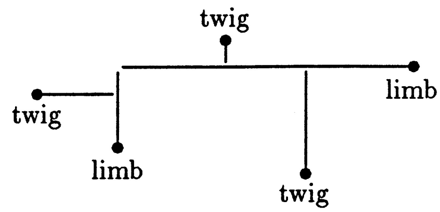

Figure 4: A bough with an exceptional Steiner point.

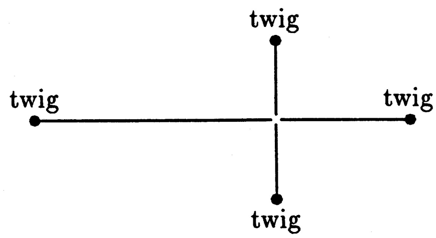

Figure 5: A degenerate bough.

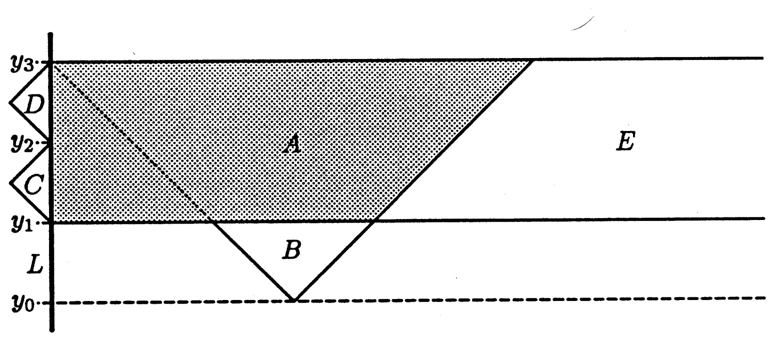

Figure 6: Regions involved in an obstruction to a Steiner line.