Constrained Integer Approximation to 2-D Line Intersections

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An algorithm is presented for finding the point with integer-valued coordinates that is nearest to the intersection of two lines, under the constraint that the approximating point lies in a pre-selected sector formed by these lines. The desired point can be determined in $O(log_2N)$ time in the worst case, where N is the largest integer among the numerators and the denominators of the rational numbers representing the slopes of the lines.

The problem arises in using finite-precision arithmetic to compute the intersection of polygons in such a way that the computed intersection is *guaranteed* to be contained in the true intersection. In multiple polygon intersections, computed vertices at each stage are used to compute vertices at the next and subsequent stages. If floating-point computation is used, accumulated round-off error through all previous stages render the location of the final vertices totally unpredictable.

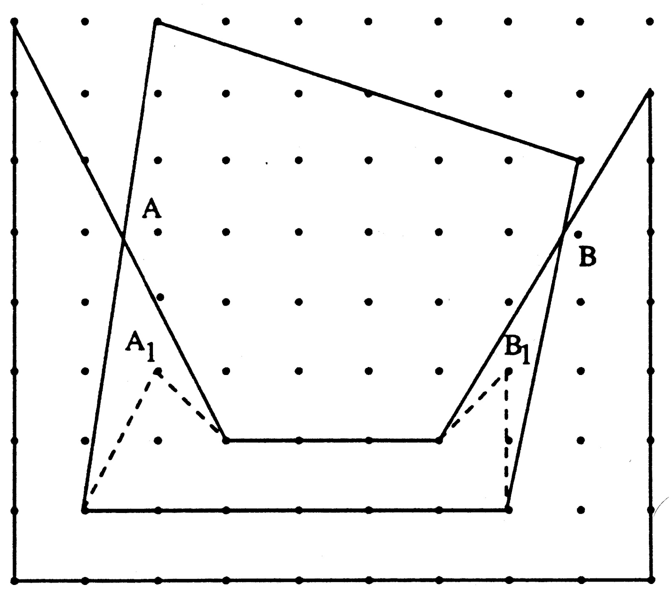

Figure 1: Approximation of polygon intersection on the integer grid. Points A_1 and B_1 are the approximations of A and B respectively.

The intersection of lines specified by integer-valued parameters, such as a pair of points, may be computed *exactly* as a pair of rational coordinates. This "rational-point" may then be approximated by an integer-point (a point on the integer-grid). If the rational-point is a new vertex of an

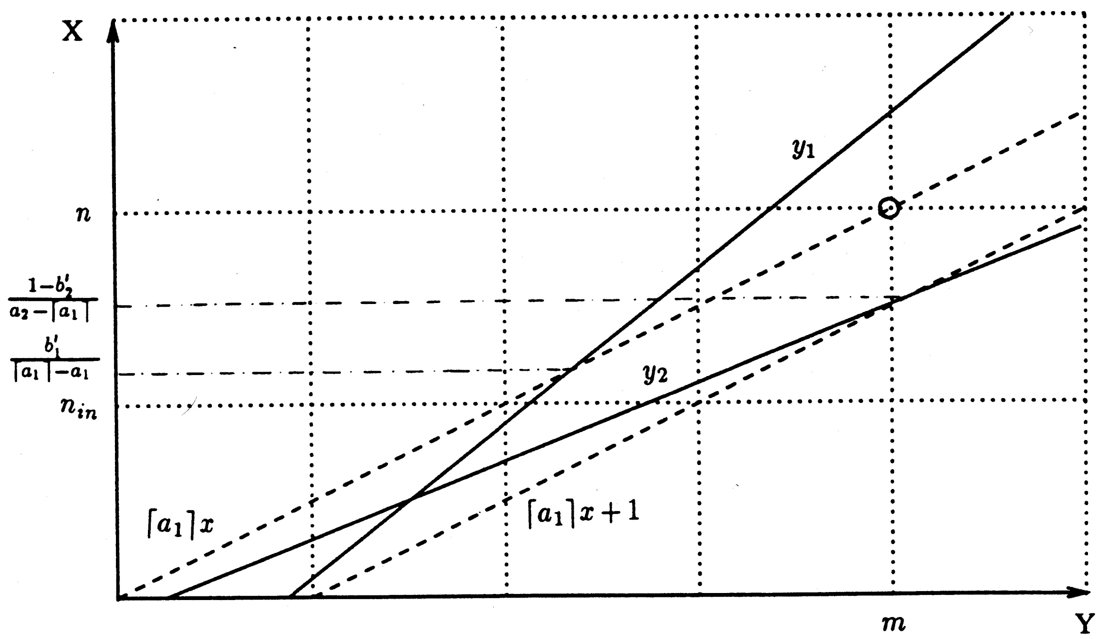

Figure 2: Example 1.

intersection polygon, it is necessary to locate the approximating point inside the polygon (Figure 1). Hence we must find the grid point *closest* to the intersection but *inside* the appropriate sector formed by the edges of the intersection polygon.

When the specified sector subtends the wider angle ($\geq \pi/2$) at the intersection, then the approximating point may be found readily. In this case the desired point is one of the four vertices of the grid cell containing the intersection point.

The case of the narrower sector ($\leq \pi/2$) is in general non-trivial. Since the integer grid maps to itself when rotated by $\pi/2$, we will consider, without loss of generality, only the case where the slope of one of the lines is between 0 and ∞ and the slope of the second line is between $-\infty$ and ∞.

If the intersecting lines straddle a line with integer-valued slope, then the approximating point can be obtained in one step. Since our strategy will be to reduce the general problem to this configuration, we give the explicit solution. The proof is suggested by Figure 2. Straddling the horizontal axis (one line with positive and the other with negative slope) is a special case of this condition.

THEOREM 1. *Given $y_1 = a_1x + b_1$ and $y_2 = a_2x + b_2$ such that $\lceil a_1 \rceil \leq a_2$ but $a_1 \neq a_2$. If n_{in} is the smallest integer such that $y_1(n_{in}) \leq y_2(n_{in})$ then n is the smallest integer such that the interval $[y_1(n), y_2(n)]$ contains an integer, where*

$$n = \begin{cases} n_{in}, & \text{if } 1 \leq b'_2 \\ n_{in} + \min\left\{ \left\lceil \frac{b'_1}{\lceil a_1 \rceil - a_1} \right\rceil, \left\lceil \frac{1-b'_2}{a_2 - \lceil a_1 \rceil} \right\rceil \right\}, & \text{otherwise.} \end{cases}$$

The values of n_{in}, b'_1, and b'_2 are
$n_{in} = \lceil (b_1 - b_2)/(a_2 - a_1) \rceil$, $b'_1 = b_1 + n_{in}a_1 - \lfloor (b_1 + n_{in}a_1) \rfloor$, $b'_2 = b_2 + n_{in}a_2 - \lfloor (b_1 + n_{in}a_1) \rfloor$.

EXAMPLE 1: Given $y_1 = (5/4)x + 7/8$ and $y_2 = (5/2)x + 1/4$, from Theorem 1, $n_{in} = 1$, $b'_1 = 1/8$, $b'_2 = 3/4$. So $n = 2$ and $m = 4$ (Figure 2).

Now we consider the only case of interest that remains, when the slopes of the lines are strictly between two consecutive integers, i.e., when $0 < \lfloor a_1 \rfloor < a_1 < a_2 < \lceil a_1 \rceil$.

EXAMPLE 2: One such configuration is $y_1 = (47/14)x + 1/4$ and $y_2 = (41/12)x + 1/24$. We wish to find the smallest integer n for which there exists an integer m such that

$$(47/14)n + 1/4 \leq m \leq (41/12)n + 1/24. \tag{1}$$

We will *reduce* such problems to the configuration in Theorem 1 by iterating two steps.

1. If slopes a_1 and a_2 are both greater than 1, subtract $\lfloor a_1 \rfloor \cdot n$ from the inequality. The resulting slopes will be in $[0, 1]$. This step does not alter the desired value of n, but the new value of m will be reduced by $\lfloor a_1 \rfloor \cdot n$.
2. Invert the inequality, which interchanges the roles of m and n. This also inverts the slopes, which will now be greater than 1. If the condition of Theorem 1 is satisfied, then solve for current m and n and stop, otherwise go to step 1.

We illustrate the above steps for Example 2:

First iteration:

Subtract $\lfloor 47/14 \rfloor \cdot n = 3 \cdot n$ from (1)

$$(5/14)n + 1/4 \leq m' \leq (5/12)n + 1/24, \quad (\text{where } m' = m - 3n)$$

Invert the inequality

$$(12/5)m' - (1/24)\cdot(12/5) \leq n \leq (14/5)m' - (1/4)\cdot(14/5) \tag{2}$$

Second iteration:

Subtract $\lfloor 12/5 \rfloor \cdot m' = 2 \cdot m'$ from (2)

$$(2/5)m' - (1/24)\cdot(12/5) \leq n' \leq (4/5)m' - (1/4)\cdot(14/5), \quad (\text{where } n' = n - 2m')$$

Inverting the inequality results in

$$(5/4)n' + (1/4)\cdot(14/5)\cdot(5/4) \leq m' \leq (5/2)n' + (1/24)\cdot(12/5)\cdot(5/2)$$
$$\text{or } (5/4)n' + 7/8 \leq m' \leq (5/2)n' + 1/4$$

Theorem 1 is applicable here and in Example 1 we have already found that $m' = 4$ and $n' = 2$. By retracing our steps, we find the solution to the original problem to be $m = 34$ and $n = 10$.

The solution can be expressed in terms of continued-fraction expansions. Note that

$$47/14 = 3 + 1/(2 + 1/(5/4)), \text{ and } 41/12 = 3 + 1/(2 + 1/(5/2)),$$

where the final quotients correspond to the slopes in the reduced problem. This suggests the following theorem.

Definition: $\langle x_1, x_2, \ldots, x_k, p'/q' \rangle$ represents the rational number $p/q = x_1 + 1/(x_2 + 1/(\ldots(x_k + q'/p')\ldots))$, where x_j, p, q, p', and q' are non-negative integers.

Theorem 2. *Let $a_1 = p_1/q_1 = \langle x_1, \ldots, x_k, p_1'/q_1' \rangle$ and $a_2 = p_2/q_2 = \langle x_1, \ldots, x_k, p_2'/q_2' \rangle$ such that $0 \le p_1'/q_1' \le \lceil p_1'/q_1' \rceil \le p_2'/q_2'$ but $p_1'/q_1' \ne p_2'/q_2'$. If m' and n', given by Theorem 1, are the smallest integers satisfying*

$$(p_1'/q_1')n' + (-1)^k(q_1/q_1')b_1 \;\le\; m' \;\le\; (p_2'/q_2')n' + (-1)^k(q_2/q_2')b_2,$$

then m and n given by $m/n = \langle x_1, \ldots, x_k, m'/n' \rangle$ are the smallest integers such that

$$(p_i/q_i)n + b_i \;\le\; m \;\le\; (p_j/q_j)n + b_j, \quad \text{where } p_i/q_i \le p_j/q_j.$$

Note: m and n are, respectively, the numerator and the denominator of $\langle x_1, \ldots, x_k, m'/n' \rangle$ when it is evaluated without cancellation of any common factor.

From the definition, we have $47/14 = \langle 3, 2, 5/4 \rangle$ and $41/12 = \langle 3, 2, 5/2 \rangle$. From Theorem 1, $m' = 4$ and $n' = 2$ satisfy

$$(5/4)n' + (-1)^2(1/4)\cdot(14/4) \;\le\; m' \;\le\; (5/2)n' + (-1)^2(1/24)\cdot(12/2).$$

Since $\langle 3, 2, 4/2 \rangle = 34/10$, according Theorem 2 $m = 34$ and $n = 10$ are the smallest integers that satisfy

$$(47/14)n + 1/4 \le m \le (41/12)n + 1/24.$$

To evaluate the time complexity of the algorithm, we assume that the numerators and the denominators of the rationals in the slopes and the intercepts are within the range of integer representation N of the machine. Integer division and modulo arithmetic can therefore be performed in constant time. Then the time required to find the approximating point is proportional to the number of steps in the continued-fraction expansion of the two slopes (up to the point where the two expansions differ for the first time) - all other parts of the algorithm can be executed in constant time.

The continued-fraction expansion of a rational p/q has at most $2 + 2log_2 \min\{p, q\}$ steps, so it is bounded by $2 + 2log_2 N$. The worst case occurs when the two expansions do not differ until one of them is fully expanded. Thus the worst-case time complexity of the algorithm is bounded by $O(log_2 N)$.

In general, however, the algorithm executes much faster because in the majority of the cases the slopes are separated by an integer and no continued-fraction expansion is required. For slopes uniformly distributed over the range achievable with numerator and denominator bounded between 0 and N, the average complexity is $O(1)$. We conclude that the above method provides an efficient solution for applications that require constrained finite-precision approximation.

The formal proofs have been developed and the algorithm has been coded and tested.