Ham-Sandwich Sectioning of Polygons

Matthew Díaz* Joseph O'Rourke†

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Abstract

The classic ham-sandwich theorem in two dimensions states that given two finite sets of points B and H (Bread and Ham), there exists a line that simultaneously bisects both sets. This paper examines a continuous version of this process: sectioning polygons according to area rather than according to the cardinality of a finite set of points. We consider three types of sectioning of increasing complexity: bisections, ham-sandwich cuts, and orthogonal four-sections. For convex polygons algorithms exist [Sto89] for determining bisectors and ham-sandwich cuts in linear time. For simple polygons we present a straightforward $O(n \log n)$ bisection algorithm. This is then used in a numerical search to obtain an $O(nKG \log n)$ algorithm for determining the ham-sandwich cut of two simple polygons, where G is the number of input bits needed to express the coordinates of the polygon and K the number of output bits of accuracy in the solution partition.

An orthogonal four-section of a polygon is a pair of orthogonal lines that partition the area into four equal parts. We seem to be the first to explore this type of sectioning. We first obtain an $O(n^2)$ algorithm for finding orthogonal four-sections of finite point sets, and then extend this to the continuous case, using a continuous analog of the median level in a dual arrangement. This algorithm runs in time $O(n + R)$ where R is the time (independent of n) of finding the roots of a certain polynomial of one variable of fixed degree. We prove that such a section always exists, but we display examples where it is not unique.

1 Introduction

Given a finite set of points B there always exists a line L with the property that the number of points of B in each of the open half-planes determined by L are equal. In fact, such a line exists for any arbitrary orientation. Given two finite point sets B and H the ham-sandwich theorem [Ede87, p. 69] states that there always exists a line that simultaneously bisects B and H. Algorithms exist [Ede87, p. 344] that will determine the ham-sandwich cut for two point sets in time $O(n + m)$, where n and m are the cardinality of B and H, respectively.

We begin with a discussion of algorithms that bisect the area of a polygon given either an orientation or a point on the hull. These are then used as the basis for a numerical search algorithm that finds the ham-sandwich cut of two line-separable polygons.

We next examine the notion of an *orthogonal* four-section – a pair of mutually perpendicular lines that partition a point set or polygon into four equal regions. We derive properties of such a four-section and present algorithms for its determination given a discrete point set or convex polygon. We conclude with a discussion of work in progress and future directions. This extended abstract is a condensed version of [DO90] and as such many details and proofs have been omitted. The reader should consult the full paper for more information.

*Dept. of Computer Science, The Johns Hopkins University, Baltimore, MD 21218

†Dept. of Computer Science, Smith College, Northampton, MA 01063

2 Area Bisection of a Polygon

Given a polygon P and an angle θ we denote by $L(\theta)$ an arbitrary line which forms an angle θ with the positive x axis, by $L_2(\theta, P)$ the line with orientation θ which bisects the area of P, and by $L_2(p, P)$ the line through point p which bisects the area of P. We are guaranteed of the existence and uniqueness of both $L_2(\theta, P)$ and $L_2(p, P)$.

In [Sto89] algorithms are presented for determining these bisectors for convex polygons in time $O(n)$. For simple polygons we have shown [DO90]:

Lemma 2.1 *Given a polygon P with n vertices, an orientation θ, and a point p on the convex hull of P, the bisector $L_2(\theta, P)$ or $L_2(p, P)$ can be found in time $O(n \log n)$.*

3 Ham-Sandwich Cuts

Two polygons are said to be line-separable if there exists a line that separates the polygons. Given two line-separable polygons P_1 and P_2 we wish to find their ham-sandwich cut – a line that simultaneously bisects the area of both polygons.

We begin with the following two lemmas:

Lemma 3.1 *Given a polygon P and bisector $L_2(\theta, P)$, let a and b be the intersection of $L_2(\theta, P)$ with the convex hull of P. Then as θ varies from 0 to π, a and b progress monotonically counterclockwise about the hull of P.*

Lemma 3.2 *Given two line-separable polygons, P_1 and P_2, the ham-sandwich cut, $L_{hs}(P_1, P_2)$, is unique.*

Note that Lemma 3.2 does not hold for non-line-separable polygons, as examples exist of pairs of polygons with more than one ham-sandwich cut

For convex polygons an algorithm exists [Sto89] for finding the ham-sandwich cut of two line-separable polygons in $O(n)$ time. This algorithm works by quickly isolating the edges through which the ham-sandwich cut passes, and then performing a calculation to determine the exact orientation. For non-convex polygons this calculation proves too unwieldy for practical use, and so we turn to a more practical, but approximate, numerical search algorithm. Given two line-separable polygons P_1 and P_2 it proceeds by finding for a given orientation θ the bisectors $L_2(\theta, P_1)$ and $L_2(\theta, P_2)$. A search is then performed over θ until the intersection points of the bisectors with the hulls become collinear. Lemma 3.1 guarantees that these hull intersection points permit a search, while Lemma 3.2 provides assurance that the search will converge correctly, resulting in:

Lemma 3.3 *For a pair of line-separable polygons with a total of n vertices, the orientation of the ham-sandwich cut can be determined in time $O(nKG \log n)$, where K represents the desired number of bits of accuracy in the orientation of the ham-sandwich cut and G the number of bits needed to express the input coordinates.*

Note that when applied to convex polygons, this algorithm runs in time $O(nKG)$.

4 Orthogonal Four-sections

For a set of n points B, a four-section of B is a pair of lines such that the number of points in each of the open wedges formed by these lines is no greater than $\lceil n/4 \rceil$. For a polygon P, we require that each wedge contain one fourth of the area. An arbitrary four-section can be determined by first picking a bisector and then finding the ham-sandwich cut of the two regions formed; these two regions are line-separated by the bisector, and so Lemma 3.3 applies. Here we investigate orthogonal four-sections – a four-section generated by two mutually orthogonal lines. The existence of such a section is guaranteed by the following lemma.

Lemma 4.1 *Given a set of points B, there always exists two perpendicular lines that divide B into four equal regions (B is four-sectioned). The existence of an orthogonal four-section is also guaranteed for a simple polygon P.*

It is interesting to note that while there must always exist an orthogonal four-section, there exist examples of polygons (e.g., any centrally symmetric figure) such that the *only* four-section is orthogonal. For a set of points it is not known whether analogous examples exist.

4.1 Orthogonal Four-section of a Point Set

The first step in the determination of an orthogonal four-section for a set of points B is to find a representation of all of the combinatorial classes of bisectors for B. As both lines in an orthogonal four-section are bisectors, given such a representation the problem can then be reduced to a search to find a pair of bisectors which four-section and are mutually orthogonal.

One such representation is provided by *k-levels* within the arrangement of lines A_B formed through a dual transformation of the points in B [Ede87, p. 48]. In particular the $n/2$ or median level of this arrangement is composed of a set of segments and rays each representing the combinatorial class of bisectors of B that pass through a point of B.

The algorithm works by spinning a pair of mutually orthogonal bisectors through the point set, using the median level in A_B to coordinate the search. A set of counters are initialized and maintained to keep track of the number of points in each of the four wedges. The search continues until a four-section is found. The complexity of the algorithm is dominated by the construction of the arrangement [EOS86], leading to

Lemma 4.2 *Given a set of n points B, an orthogonal four-section can be determined in time $O(n^2)$.*

4.2 Orthogonal Four-section of a Convex Polygon

In the continuous domain, finding an orthogonal four-section of a polygon proves to be a greater challenge. While many of the techniques used for the discrete algorithm can be generalized, maintaining track of the area in each of the wedges can no longer be accomplished with a set of counters, but must be implicitly kept through a functional description. It is this aspect of the algorithm which, though resolved theoretically, makes it somewhat impractical to implement.

The first step in generalizing the discrete algorithm is to find a continuous analog to the median level. In the discrete case the median level represented the different combinatorial classes of bisectors for the set, and was conveniently obtained from an arrangement arising from the dual transformation of the point set. For the case of a convex polygon P with n vertices, while we have no such arrangement to use as a basis, the concept of a median level is still valid. For each bisector of P we have a point in the dual, and since for each possible orientation we know a unique bisector must exist, the set of all bisectors of P gives rise to a set of points in the dual space. It can be shown that this set forms in fact a continuous curve. We refer to this curve as the ***median curve***.

Given an initial bisector with orientation θ, the ham-sandwich cut is uniquely determined. As we vary θ the orientation ω of the associated ham-sandwich cut varies also. If we restrict our attention to the situation were the pair of edges on which the bisectors lie is fixed, then we have the following characterization:

Lemma 4.3 *Let $\omega = f(\theta)$ represent the functional relationship between the bisector and ham-sandwich cut along the median curve. Then over the range of input such that the two lines touch the same two pairs of edges, the function f is a polynomial of bounded degree in* $\cos(\theta)$.

Using the same search paradigm as in the discrete case, we isolate the four edges of the polygon through which the orthogonal four-section passes in $O(n)$ time. We then solve equation f from Lemma 4.3 in time $O(R)$ to determine an orthogonal four-section, yielding

Lemma 4.4 *Given a convex polygon P with n vertices, an orthogonal four-section can be determined in time $O(n+R)$.*

It should be noted again that due to the complexity of the functions involved in Lemma 4.3, this algorithm would not lend itself to a practical implementation. An example of an orthogonal four-section is displayed in Figure 1.

For a centrally symmetric polygon there are an infinite number of orthogonal four-sections since any four-sectioning is orthogonal. An analysis and numerical search described in [DO90] resulted in discovering two distinct orthogonal four-sections for a non-symmetric polygon both of which pass through the same pairs of edges. This lack of a unique orthogonal four-section for the same pair of cut edges, dispels hope that a binary search might replace the complex computations required by Lemma 4.3.

5 Conclusion

We have presented algorithms for finding bisectors, ham-sandwich cuts and orthogonal four-sections for polygons. There are several directions for further research:

Improve to $O(n^{3/2})$. In the algorithm for the determination of the orthogonal four-section of a point set, the complexity of the median level is only $O(n^{3/2})$ while the total time is dominated by the $O(n^2)$ required for the arrangement construction. It may be possible to bypass the construction of the entire arrangement and instead form only the needed portions, thereby lowering the complexity.

General Polygons. The algorithm for finding the orthogonal four-section of a polygon is limited to convex polygons due to our lack of knowledge of the functional relationship between the two bisectors for general polygons. An analysis that encompasses the possibility of a bisector intersecting a linear number of edges needs to be performed to see if this algorithm can be extended.

Three Dimensions. Preliminary work generalizing this search method to three dimensions shows promise of finding an orthogonal eight-section of a set of points in faster than brute force time.

References

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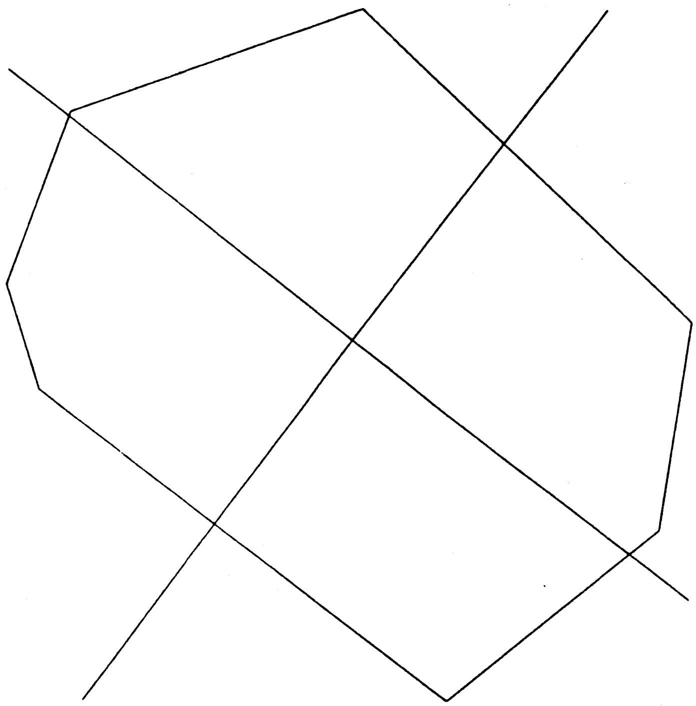

Figure 1: Orthogonal four-section of a convex polygon.

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