Optimum Watchmen Routes in Spiral Polygons: Extended Abstract *

Bengt J. Nilsson† Derick Wood‡

1 Introduction

The problem of placing guards in an art gallery so that every point in the gallery is visible to at least one guard has been considered by several researchers. The gallery is represented by a polygon (having n vertices) and the guards are points in the polygon. Chvàtal [Chv75] and Fisk [Fis78] proved that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary. Many other results on art gallery problems and general visibility problems can be found in [O'R87]. We consider two related problems where the first can be stated as follows.

> Given an art gallery, determine the shortest route within the gallery for a mobile guard or *watchman* that ensures each point in the gallery to be visible from at least one point on the route.

This problem has been studied by Chin and Ntafos in [CN86] and [CN88]. They prove that the problem is NP-hard for polygons with holes and for 3-dimensional simple polyhedra. They also give algorithms to solve the problem for two particular cases of polygons; an $O(n)$-time algorithm for rectilinear monotone polygons and an $O(n \log\log n)$-time algorithm for simple rectilinear polygons.

The second problem is a more general version of the first problem:

> Given an art gallery and an integer $m \geq 1$, compute the routes for m watchmen so that each point in the gallery is seen by at least one watchman from some position on his route, and the sum of the lengths of the routes is minimized.

The problem of finding routes for watchmen in polygons has only been studied in the cases when the watchmen are moving either along an edge of the polygon or along an interior line segment connecting two vertices (see [O'R87]). We will consider arbitrary watchmen.

In the next section we present and prove the correctness of an $O(n)$-time algorithm to find a shortest watchman route in spiral polygons. In Section 3 we extend the algorithm to find the routes for m watchmen in a spiral polygon so that the sum of the lengths of their routes is minimized. This algorithm uses $O(n^2m)$ time and $O(n^2)$ storage

At first sight, spiral polygons define a highly restricted class of polygons that are of little general interest. This is, however, not the case. Spiral polygons form the first level of a hierarchy of simple polygons, the so called k-spiral polygons; that is polygons having k reflex chains. This hierarchy contains all simple polygons, hence, viewed in this light our results are a first step in understanding simple polygons from the k-spiral viewpoint. It is our hope, that the techniques discussed here should be generalizable to 2- and higher order spiral polygons.

*This work was supported under a Natural Sciences and Engineering Research Council of Canada Grant No. A-5692 and under a grant from the Information Technology Research Centre. The research of the first author was done while visiting the University of Waterloo.

†Department of Computer Science, Lund University, Box 118, 221 00 Lund, Sweden

‡Department of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

2 Finding a Shortest Watchman Route

A simple polygon P is a *spiral* if its boundary can be divided into a reflex chain R and a convex chain C. A chain is *reflex* (*convex*) if its vertices are all reflex (convex) with respect to the interior of the polygon.

We define a *watchman route* of a polygon to be a simple polygonal curve inside the given polygon such that every point in the polygon is visible from at least one point on the curve.

2.1 The Geometry of Routes

The algorithm we present relies on a number of observations about spiral polygons which are formalized in the following results.

To simplify the following discussion we assume that the polygon P is not starshaped. If it is, any point in the kernel of P would be a shortest watchman route.

Extending the first and last edges of the reflex chain until they intersects the convex chain gives two line segments that we call the *essential line segments*.

LEMMA 2.1 *Any route intersecting the two essential line segments in a spiral polygon is a watchman route.*[1]

Therefore the shortest watchman route is the shortest path between the essential line segments.

The next lemma tells us where to find the intersection points of the route and the essential line segments.

LEMMA 2.2 *The shortest path between two line segments, possibly obscured by a reflex chain, is a chain such that the first and last links of the chain are either perpendicular to the line segments or connect to one of the end points.*

We are now ready to describe the algorithm for the problem of finding shortest watchman routes.

2.2 Algorithm

The idea of the algorithm is to find the shortest path between the essential line segments; by Lemma 2.1. This is done by computing the first and last links of the resulting path and then adding the intermediate links which are edges of the reflex chain; see Lemma 2.2.

To find the first link the algorithm scans the vertices of the reflex edge and computes for each vertex its perpendicular intersection with the essential line segment. If the intersection is not on the segment connect the link to the closest end point of the essential line segment. Continue scanning for the last link contained in P. This link is the first link of the chain.

This scheme will work for the general case spiral polygons but not for some exceptional cases. For instance if the spiral polygon is starshaped, then any point in the kernel will do as the shortest watchman route.

The other special case is when the essential lines *converge* by which we mean that the intersection points of the two essential line segments with the convex chain see each other. This special case has three possible routes.

- The line between the end points.
- The path from an end point that joins perpendicularly to the other essential line segment.
- The path between the the essential line segments as computed above.

[1]We omit the proofs of all Lemmas and Theorems throughout this extended abstract.

Finding the essential line segments and the first and last links takes linear time since the algorithm scans through the vertices of the polygon, spending only constant time per vertex. The special case of spiral polygons can be tested for and a shortest route can be computed in constant time given the essential line segments. The other special case can be handled in linear time since it requires at most a constant number of polygonal intersection tests.

This gives us the main result of the section.

THEOREM 1 *There exists an algorithm to compute the shortest watchman route in linear time.*

3 Finding Optimum Watchmen Routes

In this section we will extend the algorithm of Section 2 to compute m watchmen routes so that the sum of the lengths of the routes is minimized.

We need to change our definition of a watchman route from the one given in Section 2. In this section it is no longer necessary for a watchman route to see the interior of the polygon. However, every point in the polygon must be visible from some point in the *set* of routes that we call the *watchmen routes*.

We define the set of *all essential line segments* to be the the set of line segments that are the extensions of the edges of the reflex chain. Each extension defines two essential line segments, the *forward* and *backward* essential line segments.

The idea of the algorithm is to choose $m-1$ cut points so that we can find the complete set of routes by computing the shortest watchman route from every cut point of the polygon to the next. A *cut point* is a vertex of the reflex chain. The interior bisector from a cut point intersects the convex chain and divides (cuts) the original spiral polygon into two new ones; either both spiral or one spiral and one convex. Dividing the watchmen between the two sub-polygons we now have two smaller independent versions of the original problem. Thus we solve each subproblem recursively by the same method. To find the optimum set of routes the algorithm has to try all possible cut points.

Since this recursive scheme would yield an exponential time algorithm we need to be more clever in approaching the problem. We use the *dynamic programming* technique to improve the time bounds of our algorithm at the cost of using more storage.

3.1 Properties of Watchmen Routes

We begin this section with a discussion of the properties of watchmen routes. We notice that it is only necessary to consider visibility of edges of the reflex chain, a property that we make extensive use of.

LEMMA 3.1 *A collection of guards (mobile or stationary) see the spiral polygon if and only if they see all edges of the reflex chain.*

The next Lemma proves the correctness of our algorithm. It shows that by trying all possible cut points of a spiral polygon we must necessarily end up with the optimum watchmen routes.

LEMMA 3.2 *The optimum watchmen routes are a sequence of shortest watchman routes each starting at a forward essential line segment and ending at a backward one. In between such routes there is exactly one cut point.*

3.2 The Optimum Watchmen Routes Algorithm

The idea of the algorithm is to precompute all single shortest watchman routes as given in Section 2 between all the essential line segments defined by the edges of the reflex chain. The set of watchman routes is stored in a matrix. The main step of the algorithm is then to try all possible cut points and put one shortest watchman route in the left subpolygon and $m-1$ routes in the right subpolygon. The problem is thus divided into one single shortest watchman route problem for which the result is already stored in our matrix, and a smaller subproblem of size $m-1$.

3.2.1 Preprocessing

The preprocessing step computes the shortest paths between all forward essential line segments and backward essential line segments. Since there exists, in the worst case, a linear number of essential line segments there could exist a quadratic number of shortest paths. We can easily produce a cubic time algorithm to solve the problem, but since we are aiming for a better than cubic time bound we need to be more clever. Instead of computing complete routes, we only compute the first and last links of the routes from which it is easy to get the length of that route. By doing incremental computations from one forward essential line segment to all backward ones, following the forward one and using the previous route to compute the next, it is possible to achieve a quadratic time algorithm for this problem.

3.2.2 The Main Routine

The dynamic programming algorithm finds the optimum watchmen routes for m watchmen by computing the watchmen routes from the last backward essential line segment to all forward ones for all numbers of watchmen from 1 to m.

The optimum watchmen routes from some forward essential line segment to the backward essential line segment of the last edge of the reflex chain with i watchmen is obtained from a shortest watchman route from the starting forward essential line segment to some backward one. The route is followed by the watchmen routes of $i-1$ routes from the next edges forward essential line segment to the last backward one.

For each of the i watchmen routes the algorithm does at most a linear number of work (to test for the optimum routes). Since the i watchmen routes are computed from all forward essential line segments and for all values of i between 1 and m we have an $O(n^2m)$ time bound for the algorithm. The storage bound is $O(n^2)$ since we need quadratic storage for all the shortest watchman routes and $O(nm)$ for all the i watchmen routes.

This gives us our main result.

THEOREM 2 *The optimum watchmen routes of a spiral polygon can be computed in $O(n^2m)$ time and $O(n^2)$ amount of storage.*

References

[Chv75] V. CHVÀTAL. *A Combinatorial Theorem in Plane Geometry.* Journal of Combinatorial Theory B, 13(6):395–398, 1975.

[CN86] W. CHIN, S. NTAFOS. *Optimum Watchman Routes.* In Proc. Sec. ACM Symposium on Computational Geometry, pages 24–33, 1986.

[CN87] W. CHIN, S. NTAFOS. *Shortest Watchman Routes in Simple Polygons.* Technical report, Dept. of Computer Science, Univ. of Texas at Dallas, 1987.

[CN88] W. CHIN, S. NTAFOS. *Optimum Watchman Routes.* Information Processing Letters, 28:39–44, 1988.

[Fis78] S. FISK. *A Short Proof of Chvàtal's Watchman Theorem.* Journal of Combinatorial Theory B, 24:374, 1978.

[O'R87] J. O'ROURKE. *Art Gallery Theorems and Algorithms.* Oxford University Press, 1987.