Linear-Time Algorithms for Weakly-Monotone Polygons

(extended abstract)

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Abstract

We introduce a new class of simple polygon, *weakly-monotone*, and give an optimal triangulation algorithm for the class. We also present a simple linear-time detection algorithm, which for input polygon P returns the set of directions in which P is weakly-monotone.

1 Introduction

Much work in computational geometry has focused on special classes of simple polygons. In this paper, we introduce to the hierarchy a new class of polygon, *weakly-monotone*, which contains the *monotone* class. For many classes of polygons, such as monotone and star-shaped, there exist linear-time algorithms for determining if a polygon belongs to the class ([PrS],[LP]). These detection algorithms are of interest for the insight they provide into the structure of polygons. In this paper we present a linear-time detection algorithm for weakly-monotone polygons, which for input polygon P returns the set of directions in which P is weakly-monotone.

A detection algorithm for a special class of polygon takes on added importance with efficient algorithms that operate on the class. For example, there exist simple, linear-time algorithms for triangulating a monotone [GJPT] or a star-shaped [ET] polygon. In the full version of this paper, we present a simple, linear-time triangulation algorithm for weakly-monotone polygons, which together with the detection algorithm allows us to triangulate a weakly-monotone polygon in linear time, without prior knowledge of the polygon's weak-monotonicity.

Of course, a weakly-monotone polygon, or a star-shaped or monotone polygon, can be triangulated directly by any of the many general polygon triangulation algorithms. However, each of the general methods has a shortcoming. The only optimal algorithm [Ch] is conceptually difficult, and too complex to be considered practical. Many of the general algorithms are simpler ([GJPT],[KKT],[To]), but each is super-linear in the worst case. In this paper we show how to triangulate a weakly-monotone polygon P, without prior knowledge of P's weak-monotonicity, with algorithms that are optimal, practical, and conceptually simple.

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2 Weakly-Monotone Polygons

Suppose we have a polygon P with vertices s and t, and let θ be a direction. Imagine two cars, one which drives clockwise along P from s to t, and the other which drives counterclockwise on P from s to t. If neither car faces direction θ during its drive, we say that P is *weakly-monotone* in direction θ for *splitting points* s and t (see Figure 1).

We now give some auxiliary definitions, and a more formal definition of weakly-monotone. Given a polygonal chain c and a direction θ, write c as the concatenation of (maximal) $(\theta + \pi)$-monotone subchains $c = c_1, \ldots, c_k$; we say that the chain c is *weakly-monotone* in direction θ if the following holds: any line in direction θ that intersects c must do so in such a way that if $p \in c_i$ and $q \in c_j$ are two points of intersection with p preceding q, then $i \leq j$. If a and b are two points of a polygon P, then $P_{CW}(a,b)$ and $P_{CCW}(a,b)$ are the subchains of P obtained by traversing P from a to b clockwise and counterclockwise, respectively (clockwise is the direction of traversal such that the interior of P lies to the right of each oriented edge of P). We say that a polygon P is *weakly-monotone* in direction θ with *splitting points* s and t if $P_{CW}(s,t)$ and $P_{CCW}(s,t)$ are weakly-monotone in direction θ. A polygon monotone in θ is clearly weakly-monotone in θ.

If we are given that polygon P is weakly-monotone in direction θ for vertices s and t, we can triangulate P in linear time. The basic idea is as follows. Assume without loss of generality that θ is the horizontal direction to the right ($\theta = 0$). If S_L is the ray with root s in direction $\theta = \pi$, and T_R the ray with root t and $\theta = 0$, append S_L and T_R to $P_{CW}(s,t)$ and $P_{CCW}(s,t)$, so that $P_{CW}(s,t)$ and $P_{CCW}(s,t)$ are infinite, simple chains. The intersection of the region below $P_{CW}(s,t)$ with the region above $P_{CCW}(s,t)$ is P. Furthermore, a partial horizontal visibility map of each region can be computed in linear time, by taking advantage of the weak-monotonicity of the boundary chain. It is not difficult to merge the maps of the regions to obtain a partitioning of P into polygons monotone in the vertical direction. These monotone polygons are in turn triangulated by the algorithm of [GJPT]. Each step is simple and runs in linear time. The algorithm can also triangulate the exterior of P. Details are given in [HM] and the full version of this paper.

There exist specific linear-time triangulation algorithms for many classes of polygons. A hierarchy of polygons is presented in [ET], where any polygon in the hierarchy can be triangulated by some specific algorithm more simple than that of [Ch]. All classes in the hierarchy are contained in the class of *crab-shaped* polygons, for which there exists no specific triangulation algorithm. Figure 2 shows that neither the crab-shaped nor the weakly-monotone class contains the other class. The class of *anthropomorphic* polygons has a simple triangulation algorithm [To]; this class neither contains nor is contained in either the crab-shaped or weakly-monotone classes. Figure 3 demonstrates the lack of inclusion.

3 Weak-Monotonicity Testing

In this section we present our main result, a linear-time detection algorithm for weak-monotonicity. We require some additional definitions.

We represent directions as polar angles measured in radians on the unit circle in the usual way. Thus, $\theta \in [0, 2\pi)$ for any direction θ. We consider a polygon P with vertices $p_0, \ldots, p_{n-1}$ ordered counterclockwise on P, and edges $e_0, \ldots, e_{n-1}$, where $e_i = \overline{p_i p_{i-1}}$ is directed from p_{i-1} to p_i (arithmetic is mod n). Let ϕ_i denote the direction of edge e_i. For a vertex p_i with incident edges e_i and e_{i+1}, the directions ϕ_i and ϕ_{i+1} partition the unit circle into two arcs, one of less than π radians and one of more than π radians. Define the *sweep closure* of p_i, $\overline{sw}(p_i)$, to be the smaller (closed) arc. For a subchain $P_{CCW}(a, b)$, $\overline{sw}(a, b) = \bigcup \overline{sw}(p_i)$, where the union is over all vertices p_i of $P_{CCW}(a, b)$ except a and b. The *sweep* of a subchain $P_{CCW}(a, b)$, denoted $sw(P_{CCW}(a, b))$ or, simply, $sw(a, b)$, is the interior of $\overline{sw}(a, b)$; that is, all directions of $\overline{sw}(a, b)$ but the two boundary directions.

With our definition of sweep, we can give an alternate definition of weakly-monotone: a simple polygon P is *weakly-monotone* in direction θ if there exist vertices s and t such that $\theta \notin sw(P_{CW}(s, t))$, $\theta \notin sw(P_{CCW}(s, t))$. If $\theta \notin sw(P_{CW}(s, t)), sw(P_{CCW}(s, t))$, then $\theta + \pi \notin sw(P_{CW}(t, s)), sw(P_{CCW}(t, s))$, so a polygon is weakly-monotone for pairs of opposite directions. Similarly, a polygon is monotone in the traditional sense for pairs of opposite directions. In fact, we can demonstrate the similarity between weakly-monotone and monotone polygons by rephrasing the usual definition of monotone polygons: a polygon P is *monotone* in directions $\theta, \theta + \pi$ if there exist vertices s and t such that $sw(P_{CW}(s, t))$, $sw(P_{CCW}(s, t)) \subset (\theta - \pi/2, \theta + \pi/2)$.

A simple polygon has two types of vertices: convex and reflex. For points a and b on P (not necessarily vertices), the *interior vertices* of $P_{CCW}(a, b)$ are all vertices of P on $P_{CCW}(a, b)$ except a and b. Given a polygon P and points a and b on P, we define $\Delta\theta(a, b)$ as the sum of the measure of the angle turned (in radians) for all convex interior vertices of $P_{CCW}(a, b)$ minus the sum for all reflex interior vertices. Basically, $\Delta\theta(a, b)$ measures the number of radians swept counterclockwise in traversing $P_{CCW}(a, b)$ from a to b.

We will now discuss an approach that partitions the boundary of P by choosing midpoints of edges of P as the partition points. In this manner every vertex of P is an interior vertex of some subchain of the partition. We call a subchain $P_{CCW}(a, b)$ a *reflex chain* if $\Delta\theta(a, b) < 0$. A subchain $P_{CCW}(a, b)$ is a *maximal reflex chain* (*mrc*) if it is a reflex chain and:

- every subchain $P_{CCW}(c, d)$ of $P_{CCW}(a, b)$ has $\Delta\theta(c, d) > \Delta\theta(a, b)$
- every superchain $P_{CCW}(c, d)$ of $P_{CCW}(a, b)$ has $\Delta\theta(c, d) \geq \Delta\theta(a, b)$.

For a reflex-chain $P_{CCW}(a, b)$, with a and b interior points of edges e_a and e_b of P, we call the vertices of $P_{CCW}(a, b)$ incident to e_a and e_b the *interior bounding* vertices of $P_{CCW}(a, b)$, and the vertices of P incident to e_a and e_b but not on $P_{CCW}(a, b)$ the *exterior bounding* vertices. In Figure 4, a maximal reflex chain $P_{CCW}(a, b)$ is shown with interior bounding vertices p_1 and p_4, and exterior bounding vertices p_0 and p_5.

Lemma 1 *The interior bounding vertices of a maximal reflex chain are reflex vertices, and the exterior bounding vertices are convex vertices.*

Lemma 2 *Maximal reflex chains do not intersect.*

Lemma 3 *Every reflex vertex belongs to a unique maximal reflex chain.*

The above lemmas establish that the entire boundary of P can be divided into alternating pieces of maximal reflex chains and convex chains, where the partitioning points are midpoints of edges of P. A maximal reflex chain may contain convex vertices, but a convex chain contains no reflex vertices. Note that for a maximal reflex chain $P_{CCW}(a, b)$, $sw(a, b)$ is an arc of $-\Delta\theta(a, b)$ radians if $\Delta\theta(a, b) \geq -2\pi$, and $sw(a, b)$ is the entire unit circle if $\Delta\theta(a, b) < -2\pi$.

In trying to show that a polygon P is weakly-monotone in a certain direction, we face the task of choosing the splitting vertices, s and t. The following lemma is the basis for our strategy of choosing splitting vertices.

Lemma 4 *Given a simple polygon P, if P is weakly-monotone in direction θ, it is weakly-monotone in θ for splitting vertices s and t that do not lie in maximal reflex chains.*

We will say that a maximal reflex chain $P_{CCW}(a, b)$ *double-sweeps* a pair of opposite directions θ, $\theta + \pi$ if $\theta, \theta + \pi \in sw(a, b)$. Note that if the orientation of a *mrc* is reversed, the double-swept pairs are unchanged. The directions double-swept by the *mrc* $P_{CCW}(a, b)$ in Figure 4 are shaded on the pictured unit circle. We can characterize the set of directions in which a polygon is weakly-monotone by the pairs that are double-swept.

Theorem 5 *A simple polygon P is weakly-monotone in the pair of directions θ, $\theta+\pi$ if and only if θ and $\theta+\pi$ are not double-swept by any maximal reflex chain.*

Proof. Suppose θ and $\theta+\pi$ are double-swept by a maximal reflex chain. Since we choose the splitting points outside of the *mrc*, one of the subchains contains the *mrc*, and therefore sweeps θ and $\theta+\pi$, regardless of orientation. Thus, P is not weakly-monotone in these directions.

Suppose no maximal reflex chain double-sweeps θ and $\theta+\pi$. Consider all vertices of P that admit tangencies in direction θ or $\theta+\pi$. Assign to each tangency either θ or $\theta+\pi$ by traversing P counterclockwise and assigning the direction encountered at that vertex. (If an edge $e_i = \overline{p_{i-1}p_i}$ faces in direction θ or $\theta+\pi$, consider p_{i-1} and p_i tangencies if p_{i-1} and p_i are both reflex or both convex.) P can be split into two subchains such that one subchain contains all θ tangencies and the other all $\theta+\pi$ tangencies.

If this were not true, we would have vertices p_i, p_j, p_k, and p_l, $i < j < k < l$, where p_i and p_k are θ tangencies and p_j and p_l are $\theta+\pi$ tangencies. Define $\Delta\bar{\theta}(p_i,p_j) = \Delta\theta(p_i,p_j) + m(\theta,\phi_{i+1}) + m(\phi_j,\theta+\pi)$, where $m(\theta_1,\theta_2) \in (-\pi,\pi)$ such that $\theta_2-\theta_1 \equiv m(\theta_1,\theta_2) \pmod{2\pi}$. If we define $\Delta\bar{\theta}$ similarly for the other three pairs, then $\Delta\bar{\theta}(\cdot,\cdot) \equiv \pi \pmod{2\pi}$ for each pair, and $\sum \Delta\bar{\theta}(\cdot,\cdot) = 2\pi$. At least one pair, say $P_{CCW}(p_i,p_j)$, has $\Delta\bar{\theta}(p_i,p_j) < 0$. By making each of p_i and p_j either an interior or exterior bounding vertex (depending on whether the vertex is reflex or convex), we obtain a reflex chain that double-sweeps θ and $\theta+\pi$, a contradiction.

If the splitting points are chosen so that the θ and $\theta+\pi$ tangencies are in separate subchains, P is seen to be weakly-monotone in θ, $\theta+\pi$. ∎

Corollary 6 *If a polygon P has a reflex chain with sweep $< -2\pi$ radians, P is not weakly-monotone.*

Corollary 7 *A polygon monotone in any direction is weakly-monotone in every direction.*

Corollary 8
A star-shaped polygon is weakly-monotone in every direction.

Theorem 5 provides us with a strategy for determining the set of weakly-monotone directions: find all maximal reflex chains, and for each *mrc* eliminate the appropriate pair of opposite arcs of directions. It is interesting to note that while the set of directions in which a polygon is monotone is a single pair of opposite arcs, there can be $O(n)$ pairs of weakly-monotone directions (see Figure 5).

The Algorithm.

We define a *turning function*, Θ_P, whose domain is the sequence of vertices encountered while twice traversing P counterclockwise, from the starting vertex p_0. (We distinguish between a vertex p_i encountered on the first traversal of P and the copy of that vertex, p_{i+n}, encountered on the second traversal.) We define $\Theta_P(p_0) = \phi_0$ and $\Theta_P(p_i) = \Delta\theta(a,p_i) + \Theta_P(p_0)$, for $i \geq 1$, where $a \in e_0$. We call the turning function the *current direction*. We also define the *front direction*, $f_P(p_i) = \max_{j=0,\ldots,i} \Theta_P(p_j)$.

The algorithm proceeds as follows. Beginning at p_0, traverse P twice, updating Θ_P and f_P at the vertices. Whenever $\Theta_P(p_i) \neq f_P(p_i)$ for the current vertex p_i, we are in a reflex chain, and if $\Theta_P(p_i) + \pi < f_P(p_i)$, then the reflex chain double-sweeps some directions. If we encounter a vertex p_j such that $\Theta_P(p_{j-1}) = f_P(p_{j-1})$ but $\Theta_P(p_j) < f_P(p_j)$ $(= \Theta_P(p_{j-1}))$, we store ϕ_{j-1} (refer to Figure 6). Upon encountering vertex p_k such that $\Theta_P(p_k)+\pi < f_P(p_k)$ $(= \Theta_P(p_{j-1}))$, we eliminate the open intervals $(\phi_k+\pi, \phi_{j-1})$ and $(\phi_k, \phi_{j-1}-\pi)$ (taken modulo 2π) as possible directions of weak monotonicity. Until $\Theta_P(p_i) > \Theta_P(p_{j-1})$ for the current vertex p_i, we retain the direction ϕ_{j-1}, enlarging the eliminated intervals until reaching a *mrc*. When once again $\Theta_P(p_i) = f_P(p_i)$, we discard ϕ_{j-1}, and thereby begin looking for the next *mrc*. We must traverse P twice because the initial vertex p_0 could be in a *mrc*.

End of Algorithm.

If (θ_1,θ_2) is a weakly-monotone interval of directions, it is possible in $O(n)$ time to find vertices s and t such that for any $\theta \in (\theta_1,\theta_2)$, P is weakly-monotone in θ with respect to splitting points s and t. Furthermore, for every (maximal) weakly-monotone interval we can compute such points s and t in $O(n)$ total time. In this way, we can preprocess P in $O(n)$ time such that, if we are given a pair of directions θ, $\theta+\pi$, we can query in $O(\log n)$ time whether this is a weakly-monotone pair, and if it is we also return a valid pair of splitting points. These query times are optimal in the sense that a polygon can have $O(n)$ pairs of opposite weakly-monotone cones, and each pair can require a distinct pair of splitting points.

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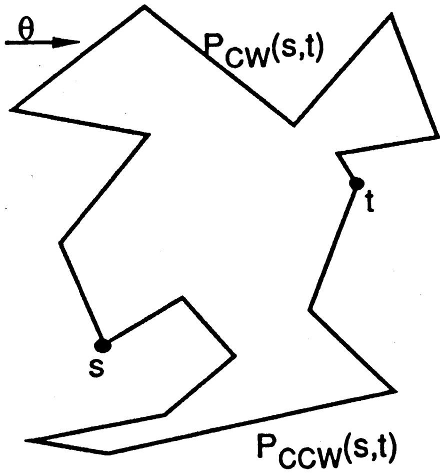
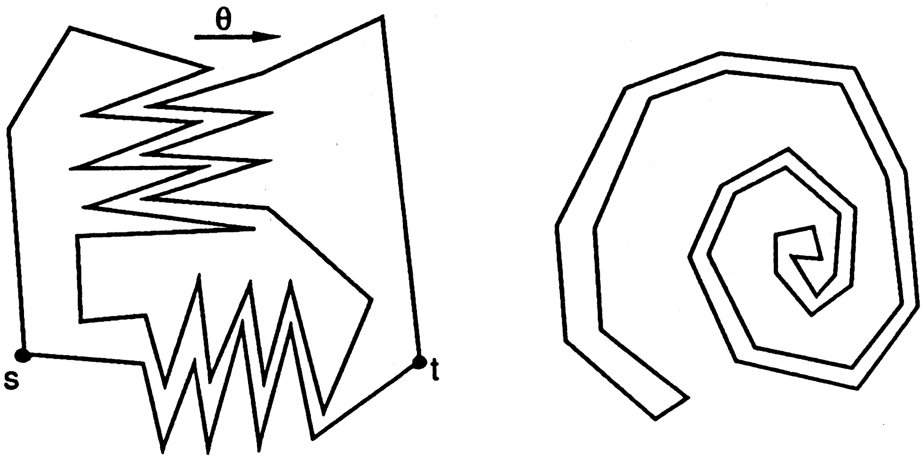


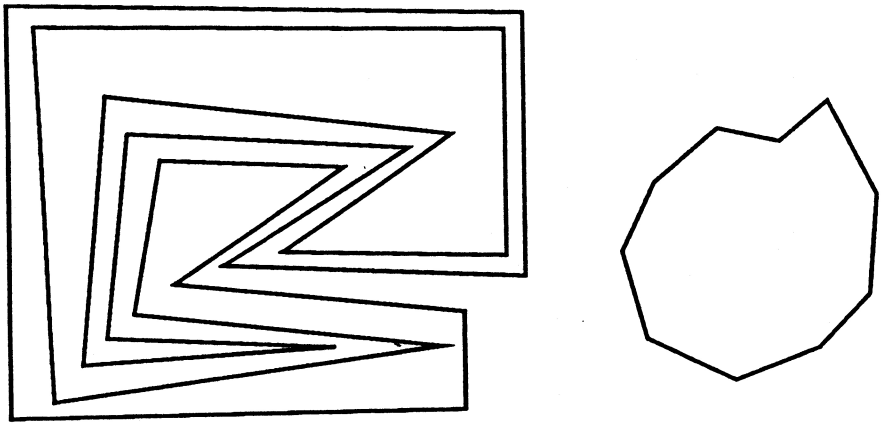
Figure 1



(a) weakly-monotone, but not crab-shaped

(b) crab-shaped, but not weakly-monotone

Figure 2



(a) anthropomorphic, but not crab-shaped or weakly-monotone

(b) crab-shaped and weakly-monotone, but not anthropomorphic

Figure 3

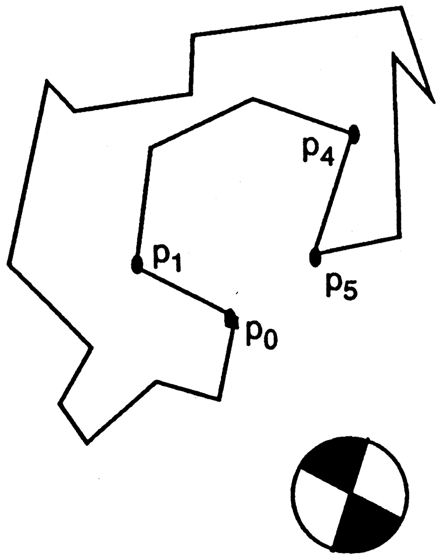


Figure 4

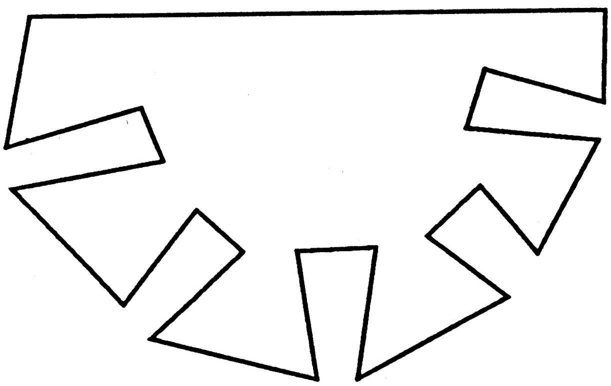


Figure 5

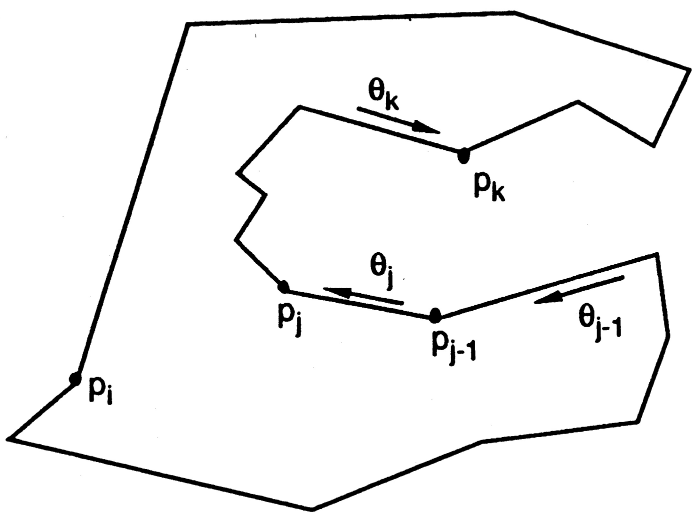


Figure 6