VORONOI DIAGRAMS COMING FROM DISCRETE GROUPS ON THE PLANE

(Extended abstract)

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0. Introduction.

Recently F. Klein [Kl] has considered the problem of constructing Voronoi diagrams for a collection of n sites in an abstract metric space M. With this generality little can be said about the Voronoi regions and imposing conditions to M or to the metric seems to be reasonable in order to make the problem more tractable.

Motivated by this observation of Klein we study in this communication the concept of Voronoi diagram in some "abstract" surfaces endowed with a distance "inter-related" with the local surface structure, obtaining the construction of these diagrams for a relevant class of such surfaces that include, among others, all the locally- Euclidean surfaces.

The same problem for specific surfaces embedded in R^3 has been solved--to our knowledge--just in two particular cases: F. Dehne and R. Klein [DeKl] showed that a sweepcircle technique for the plane can be generalized to work on the surface of a cone; and K. Q. Brown [Br] gave two different methods to construct the Voronoi diagram on the surface of a sphere. Finally we can recall that a general description of the topological and geometrical structure of the boundary of a single Voronoi region has been obtained by Ehrlich and Im Hof [EI] for the case of simply connected and complete riemannian manifolds without conjugate points.

1.The definition of a geometry. Geometries coming from a discrete group of motions of the plane.

Definition 1.1

Let M be a set and let $d:MxM \rightarrow R^+$ be defined such that:

i) $d(a,a)=0$

ii) $d(a,b)=d(b,a)$

iii) $d(a,c) \leq d(a,b)+d(b,c)$

for any a, b and c of M. d is called a <u>distance</u> in M and (M,d) is a <u>metric space</u>.

An important property of distances that could reflect the relation between the metric and the space itself is the following:

iv) For any two points a and b of M and any pair of positive real numbers α and β, there exists a collection of points $p_1, ..., p_n$ in M such that

$p_1=a, p_n=b$

$0 \leq d(p_1,p_2)+ ...+d(p_{n-1},p_n)-d(a,b)<\alpha$

$d(p_i,p_{i+1})<\beta$.

Of course this property is not always satisfyed but if d has this property we will say that the metric d satisfies the "chain condition". The collection $p_1, ..., p_n$ will be called a chain, the pairs p_i,p_{i+1} are the "links" of the chain. The length of the link p_i,p_{i+1} is $d(p_i,p_{i+1})$ and the length of the chain $p_1, ..., p_n$ is the sum of the lengths of all the links p_i,p_{i+1}.

Definition 1.2

A set M with a metric d verifying i), ii), iii) and iv) is called a geometry (following the terminology of [NiSh]).

As examples of geometries let us consider several surfaces--some of them lying in R^3, while others are more abstractly defined--which have a well known metric verifying the chain condition: the Euclidean plane, cones, cylinders, Moebius bands (all as surfaces embedded in R^3), Klein bottles and torus and proyective plane (with the metric derived from their plane covering), and, more generally, all the Riemannian manifolds, i. e. differentiable manifolds with a local inner product defined in them which is compatible with the differentiable structure. In fact, over any Riemannian manifold a metric is defined as the infimum of the lengths of piecewise differentiable paths between two given points: the local inner product allows us to compute lengths of such paths.

Now our method to construct Voronoi's on some geometries considers some canonical representation of the geometry as coming from the action on the Euclidean plane of a discrete group of motions :

Definition 1.3

A group Γ of motions of the plane is discrete if for every point A of the plane there exists a constant $c(A)>0$ depending on A, such that for every motion g in Γ with $g(A)\neq A$, it follows that $d(A,g(A))\geq c(A)$, where d here is the Euclidean distance on the plane.

Definition 1.4

Let Γ be a discrete group of motions of the plane. Two points A and A' of the plane are equivalent if one of them can be obtained from the other by a motion of Γ.

Clearly this relation is an equivalence relation and we can consider the set of equivalence classes.

Definition 1.5

If Γ is a discrete group of motions of the plane, a geometry G_Γ can be defined such that points of G_Γ are equivalence classes by Γ of points in the plane and the distance d(a,b) between two points a and b of G_Γ is defined as the shortest of the Euclidean distances d(A,B) where A runs through the set of equivalent points a and B runs through the set b. This distance is well defined and verifies i) to iv) of the definition of a geometry c.f. [NiSh].

In order to specify a point a of the geometry G_Γ we need only to know one point A of a for each equivalence class; therefore in order to determine the set of all points of the geometry G_Γ we need only to specify some region of the plane, for example a polygon, satisfying the following properties:

(1) The region contains one point from every set of equivalent points of the plane.

(2) No interior point of the region is equivalent to any other point of the region; that is, equivalent points of the region can only lie on its boundary.

Definition 1.6

A region satisfying (1) and (2) is called a fundamental domain .

From a fundamental domain for Γ, the geometry G_Γ can be considered as a surface obtained from the region by identifying or glueing together equivalent points of its boundary: it is in this way that we can identify as geometries the surfaces underlined following definition 1.2, as it can be shown that they all come (together with the considered metric) from a discrete group of motions acting on the Euclidean plane.

Therefore to construct the Voronoi Diagram for a finite collection s of points in such a surface G_Γ will mean for us to find fundamental domain D of Γ and a surjective mapping f from to the surface G_Γ and to give a partition of D such that its image by f is the Voronoi diagram in the surface G_Γ for the collection s.

2. Main results.

Let G_Γ be a geometry coming from the action of a discrete group Γ of motions of the plane and let $s=\{p_1, ..., p_n\}$ be a set of n points in G_Γ. Let D be a fundamental domain for Γ and let $S=\{P_1, ..., P_n\}$ be a set of n points in D such that $\Gamma(P_i)=p_i$. Consider the set ΓS of points in the Euclidean plane which are equivalent to some of the P_i in S.

Note that ΓS can consist of an infinite number of points but even in this case, the standard Euclidean Voronoi diagram for ΓS, Vor(ΓS), can be obtained by means of any of the algorithms that work for finite sets of points in the plane because of the following:

Theorem 1

We can explicilty give a fundamental domain D for Γ such that if we take the minimum number of copies of D: $D_0=D$, $D_1=g_1D$, ..., $D_m=g_mD$, where the g's are elements of Γ, such that their union $F=\cup D_i$ contains D in its interior and if we consider the finite set of points in the plane $S^*=S\cup g_1S\cup ...\cup g_mS$, a subset of ΓS, then $Vor(\Gamma S)=\Gamma(Vor(S^*)\cap D)$.

Moreover, knowledge of Vor(S*) suffices to construct the Voronoi diagram in G_Γ for s because of the following:

Theorem 2

The Voronoi diagram in G_Γ of s, $Vor(G_\Gamma s)$ is obtained as follows:
1^{st} take off the edges of Vor(S*) between regions of equivalent points and call V* the resulting partition
2^{nd} make the intersection of V* with D; call it W.
3^{rd} make the quotient by Γ of W.

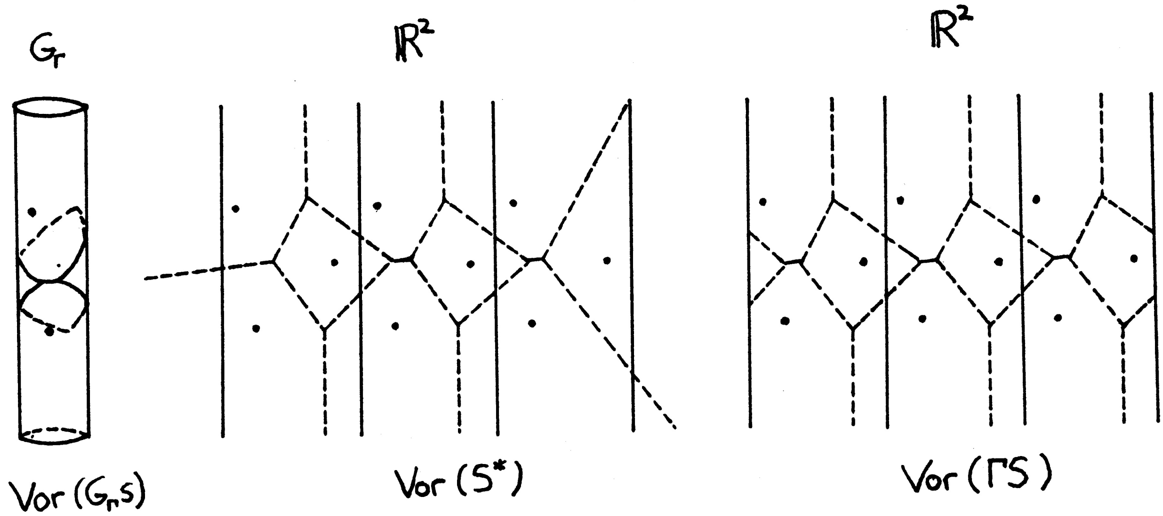
Finally let us give an example of the method outlined above:

Consider a closed band D in the plane an let Γ be the discrete group of motions generated by the traslation of a vector with origin in one edge of the band and ending in the other edge of D and perpendicular to both . The band D is then a fundamental domain for Γ and the image of D by the elements of Γ fills up the whole plane. In D, opposite points of the edges are equivalent so we can think of G_Γ as the surface of a cylinder obtained from the band D by identifying opposite points in its edges .

Let $s=\{p_1, ..., p_n\}$ be a finite collection of points on the cylinder G_Γ . The map f from D to G_Γ that takes each point of D to its equivalence class is well defined and we can find $S=\{P_1, ..., P_n\}$ a collection of points in the band D such that $f(P_i)=p_i$. Points of S in D are then repeated all over the plane by means of Γ, generating the set ΓS.

For constructing the Voronoi diagram of ΓS in the plane, we have just to compute the Voronoi diagram for the 3n points of ΓS which are in D or in the two bands adjacent to the given band D, that is Vor(S*), and to intersect it with D and repeating the pattern so obtained in D all over the plane.

For constructing the Voronoi diagram in G_Γ of s, Vor(G_Γs), we just take out from Vor(S*) the edges between regions of equivalent points, intersect with D and past opposite points in its sides.



Voronoi diagram on the surface of a cylinder

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