Polyhedra: Faces are Better than Vertices
(Extended Abstract)

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1. Introduction

A *polyhedron* is a closed surface that partitions points in E^3 into 3 sets: (i) Points lying inside the polyhedron, (ii) points incident on polyhedron and (iii) points outside the polyhedron. The boundary of the polyhedron is a two-manifold. Viewed combinatorially, the surface consists of faces, edges and vertices such that each edge is shared by exactly two faces and there is a single cycle of faces around each vertex such that each consecutive pair of faces has an edge in common. We only consider polyhedra with planar faces, that is, each face is contained in a plane. The number of holes in a polyhedron is its *genus.*

Several representation schemes have been proposed for the representation of polyhedra. Refer to [REQA80], [MANM88] for a detailed review on representations of polyhedra. Obtaining a polyhedron from its representatoin is termed *reconstruction.* If there is only a single polyhedron that can be obtained from its representation, the polyhedron is *uniquely reconstructible* in that representation scheme. The problem we consider in this paper is the unique reconstruction of polyhedra in a representation scheme encoding the *minimum* amount of information.

The spherical dual representation [ROAJ86] of a polyhedron is a *graph* in which each face of the polyhedron is a node, and is labeled by the equation of the plane containing the face. A node is connected by an arc to another if the two faces the nodes represent share an edge in the polyhedron. No ordering of the arcs around each node is specified. (We use the non-standard terms nodes and arcs for elements of the graph to avoid confusing them with the vertices and edges of the polyhedron the graph represents.) The spherical dual representation, SDR, is a representation scheme of polyhedra that can be used in both solid modeling and computer vision. A variation on this scheme of representation was used in [ROAJ87]. The SDR provides some interesting relationships between the representation of an object and its image under perspective projection [PARP89].

Edmonds theorem [GROJ87] states that every rotation system for a graph G induces (up to orientation-preserving equivalence of embeddings) a unique embedding of G into an oriented surface. Most arguments on the representation of polyhedra are based on Edmonds theorem. Using Edmonds theorem, Weiler [WEIK85] shows that knowing the ordered set of edges around each vertex or each edge, or each face of a polyhedron is sufficient information for the *unique* reconstruction of any polyhedron. Weiler also states that a representation without order has insufficient information for unique reconstruction of a polyhedron. The domain of Edmonds theorem and Weiler's work includes polyhedra with non-planar faces.

The SDR can be viewed as the dual of the wire frame representation of a polyhedron. It is well known [REQA80], [MANM88], that the wire frame representation, i.e., vertex connectivity graph of a polyhedron is ambiguous. Algorithms for the unique reconstruction of a polyhedron, given its vertex or face connectivity graph, exist only if the polyhedron has genus 0 and the graph is three-connected [HANP82].

Even though SDR is a representation devoid of order information, we give an algorithm to reconstruct any genus 0 polyhedra ***uniquely*** given its SDR. The algorithm has worst case complexity $O(n^2)$ where n is the size of the SDR. In addition we show that the SDR is not exactly the dual of the wire frame representation in the context of ambiguity. This is accomplished by an example (the 4-dimensional hypercube) which is ambiguous as a vertex connectivity graph but is unambiguous as an SDR. We then generalize this argument to prove that $SDRs$ with at most degree 4 represent polyhedra of arbitrary genus unambiguously. An $O(n)$ algorithm for unique reconstruction is also given. These results are the first of their kind.

2. Related Work

By far the most important result on the realization of a polyhedron from its adjacency graph structure is Steinitz's theorem [GRUB67]. The theorem states that *a graph G is realizable as a convex polyhedron if and only if G is planar and triconnected.* In the case of reconstruction of polyhedra, this theorem can be used for all polyhedra that are combinatorially convex. A polyhedron is called *combinatorially convex* if its vertex adjacency graph is planar and triconnected and the polyhedron has genus 0.

Given a connected graph G, a closed surface S, and an embedding $i : G \rightarrow S$, a ***dual graph*** is defined as follows. For each region f of the embedding $i : G \rightarrow S$, place a node f^* in its interior. Then, for each arc e of the graph G, draw an arc e^* between the nodes just placed in the interiors of the regions containing e. The resulting graph with nodes f^* and arcs e^* is called the *(topological) dual graph* G^* for the embedding $i : G \rightarrow S$. From Whitney, [GROJ87], a triconnected planar graph has a *unique* embedding in the plane and hence a unique dual.

Based on these theorems, [HANP82] gives a linear time algorithm for the unique reconstruction of a genus 0 polyhedron given its wire frame representation. This algorithm, however, requires the wire frame input of the polyhedron to be three-connected and planar in the graph-theoretic sense. The faces of the polyhedron correspond to the regions in the unique planar embedding of the vertex connect graph. From Whitney's theorem, every combinatorially convex polyhedron is uniquely reconstructible from its dual. This dual is the underlying graph of SDR.

Markowsky and Wesley [MARG80] present an algorithm that generates all polyhedra with a given wire frame. This explicitly uses topological and geometric information by forcing the final faces to be planar. Human intervention is required to choose a polyhedron reconstructed from such a wire frame representation.

3. Genus 0 Algorithm

Each *face* of a polyhedron consists of a connected region of a plane bounded by one or more closed circuits of *edges* (straight line segments). The end points ofan edge are the *vertices* of the polyhedron. Each line segment is shared by exactly 2 faces of a polyhedron and a vertex is common to at least 3 faces. It is possible for a pair of faces to share two or more edges. SDR does not represent such multiple adjacency.

Let $SDR = (N, A)$ be the face connectivity graph of a polyhedron P. Consider a face f_i of P and its adjacent faces $A_i = \{f_1, f_2, ..., f_k\}$. The intersection of the planes containing f and f_j, $j = 1, ..., k$, defines an infinite line l_j. Any edge shared by f_i and f_j is a line segment in l_j. The vertices of P lying in f_i are a subset of the points formed by the intersection of lines $l_1, l_2, ..., l_k$. There are $O(k^2)$ points of intersection.

Let Q_i be these intersections. Similar points are found on each plane containing a face of P. Let $Q = \cup Q_i$. A subset of these points and line segments between them constitute the actual vertices and edges of P. Many of the points in Q can be eliminated by use of the connection information in the SDR. For each point $q \in Q$, a sequence S of faces defining q is obtained from the intersections. Ordering in S is obtained by traversing across edges with q as an endpoint to an adjacent face until a cycle is complete. The definition of a polyhedron requires that there be only one circuit of polygons around a vertex. Thus for any point p that is a vertex of P, there must correspond a simple circuit of nodes in the SDR that defines p as a vertex. Accordingly, if there does not exist such a simple circuit, q is deleted from Q.

Let B_i be the set of faces forming all the points remaining in Q_i and $SDR_i = (B_i, E_i)$ be the subgraph of SDR induced by B_i. SDR_i contains all the information about face f_i since all the faces connected to f_i in both the vertex and edge sense are in SDR_i along with all the arcs in between them. Corresponding to each vertex of P in f_i, a simple circuit exists in SDR_i but the converse is not true. Each face of a polyhedron has two adjacent faces at every vertex incident to it, hence every node of SDR_i is at least of degree 2. The elimination of a node of degree two by *merging* two arcs in *series* does not affect embedding in a plane [GROJ87]. In general, no geometric information is lost since the deleted vertices of the graph are affinely dependent on the other vertices defining the region. Due to the merger of arcs in series, SDR_i may become a multigraph. Again from graph theory, we know that deletion of all but one of a set of parallel edges does not affect the embedding of the graph. No geometric information is lost since the parallel edges are only an artifact of the reduction process. We assume that SDR_i has been subject to both series and parallel reduction.

Observation 1: If SDR_i contains an articulation point, then f_i is the articulation point.

Observation 2: If SDR_i has a separation pair, f_i is in the separation pair.

Geometrically, the presence of an articulation point in SDR_i implies that the face f_i has multiple connected boundaries. In genus 0 polyhedra, the articulation point is a necessary and sufficient condition for the existence of multiple connected boundaries. A separation pair does not uniquely specify any structure in general polyhedra. In genus 0 polyhedron, however, the separation pair is a necessary and sufficient condition for the two faces representing the nodes of the separation pair to define more than one edge of the polyhedron.

If the SDR is triconnected and planar, embedding it in a plane will uniquely determine the regions and their ordering around each node. Thus a unique reconstruction is possible. The SDR and SDR_i of a genus 0 polyhedron are not necessarily triconnected. In general, the graphs have articulation points and separation pairs. Figure 1 shows a genus 0 polyhedron with an articulation point in its SDR. Finding the triconnected components of the SDR and embedding it in the plane will not work in general since some arcs are lost in decomposing the graph into triconnected components. Figure 2 shows such a polyhedron and its SDR, which on decomposition into triconnected components and embedding in the plane, defines some regions that do not correspond to any vertex of the input polyhedron. SDR_i, on the other hand, gives special status to face f_i and retains all relevant nodes connected to f_i, assuring the integrity of the embedding. The algorithm broadly consists of the following steps.

(1) For each face f_i, form the set Q_i.

(2) Form the set $Q = \cup Q_i$.

(3) Delete elements of Q that do not correspond to a circuit in SDR.

(4) For each face f_i form the subgraph SDR_i and reduce it.

(5) If SDR_i has an articulation point, decompose the graph into two-connected components, retaining the articulation point in both the components.

(6) If the SDR_i contains a separation pair, decompose the graph into three-connected components, retaining the separation pair in both components.

(7) Embed the three-connected components in the plane and find all the regions of which f_i is a part, giving all vertices formed by face f_i.

The SDR_i of each face of a genus 0 polyhedron was embedded in the plane independently to reconstruct that face of the polyhedron. In case of polyhedra of higher genus, this is not always possible. Spurious cycles are formed that cannot be detected locally. Information from other faces can be used to delete some arcs from the other SDR_i until no more arcs can be deleted. Since each loop on a face without multiple connections forms a three connected planar graph when only the connections involved in forming the loop are retained, the result of deleting arcs *ideally* results in obtaining a three connected planar graph for each loop of each face. Due to the complexity of the SDR_i of polyhedra of arbitrary genus, this reduction cannot be done for all polyhedra. The hypercube presented in the next section is an example of such an SDR. We prove that the hypercube represents a unique polyhedron.

4. The Hypercube

The 'hypercube' is the connectivity structure of the graph in Figure 3. It is the classic example of the ambiguity of the vertex connectivity graph (wire frame model) [MANM88]. The topological dual of this graph is isomorphic to itself. In contrast to the vertex connectivity representation, we show that the hypercube as an SDR is unambiguous.

Each face of a polyhedron having the hypercube as SDR is connected by an edge to four other faces. These four lines defined by the face adjacencies form two different quadrilaterals. Refer to Figure 4. Two vertices remain the same between the interpretations. One vertex a which we call the *fixed vertex* has its context unchanged, i.e., when we follow the boundary of the two polygons in the same direction, the line segments occur in the same order. At the other vertex b, called the *reflex vertex* however, the context is reversed. The internal angle at the reflex vertex changes from being a convex angle in one case to concave in the other. It is immediately clear from Figure 4 that fixing any one of the remaining four vertices $(c, d, e \text{or} f)$ determines the polygon unambiguously. Vertices on each of the lines through the fixed vertex and nearer the fixed vertex are called *intruded vertices* and those farther away are called *extruded vertices.* Thus in face 1 in Figure 4, vertices c and d are intruded vertices and e and f are extruded vertices.

For the hypercube to have two interpretations as polyhedra, some face must have two interpretations. By a long calculation, we prove the following lemma.

Lemma 1: There cannot exist a set of faces forming a reflex vertex such that every face has two interpretations.

Since only one face is sufficient to determine the polyhedron unambiguously, the following theorem immediately follows.

Theorem 2: The hypercube is ***unambiguous*** as an SDR.

The following theorem can also be proved using Lemma 1.

Theorem 3: An SDR with at most degree four unambiguously represents a polyhedron of any genus. Moreover, there is a linear time algorithm to reconstruct such a polyhedron.

5. Conjecture

We *conjecture* that the SDR is an unambiguous representation for polyhedra of arbitrary genus.

6. References

[GROJ87] Gross, J. L. and Tucker, T. W., *Topological Graph Theory*, Wiley-Interscience Publication, 1987.

[GRUB67] Grunbaum, B., *Convex Polytopes*, Interscience Publishers, 1967.

[HANP82] Hanrahan, P., "Creating volume models from edge-vertex graphs," *Computer Graphics* (Proc. of SIGGRAPH '82), vol.16(3), 1982, pp. 77–84.

[MANM88] Mantyla, M., *An introduction to Solid Modeling,* Computer Science Press, Rockville, Md, 1988.

[MARG80] Markowsky, G. and Wesley, M. A., "Fleshing out wire frames," *IBM J. Res. Develop.*, vol.24(5), 1980, pp. 582-597.

[PARP89] Paripati, P., *Polyhedra: Representation and Recognition,* M.S. thesis, Department of Computer Science, Virginia Polytechnic Institute and State University, Blacksburg, VA, 1989.

[REQA80] Requicha, A. A. G., "Representation of rigid solids - theory, methods and systems," *ACM Computing Surveys,* vol.12(4), 1980, pp. 437–464.

[ROAJ86] Roach, J. W., Wright, J., and Ramesh, V., "Spherical dual images: A 3D representation method that combines dual space and Gaussian spheres," *IEEE 1986 Workshop on Computer Vision: Representation and Control,* pp 236-241.

[ROAJ87] Roach, J. W., Paripati, P., and Wright, J., " A CAD System Based on Spherical Dual Representations," *IEEE Computer,* vol.20(8): 1987, pp. 37-44.

[WEIK85] Weiler, K., "Edge based data structures for solid modeling in a curved surface environment," *IEEE Computer Graphics and Applications*, 5(1), 1985, pp. 21–40.

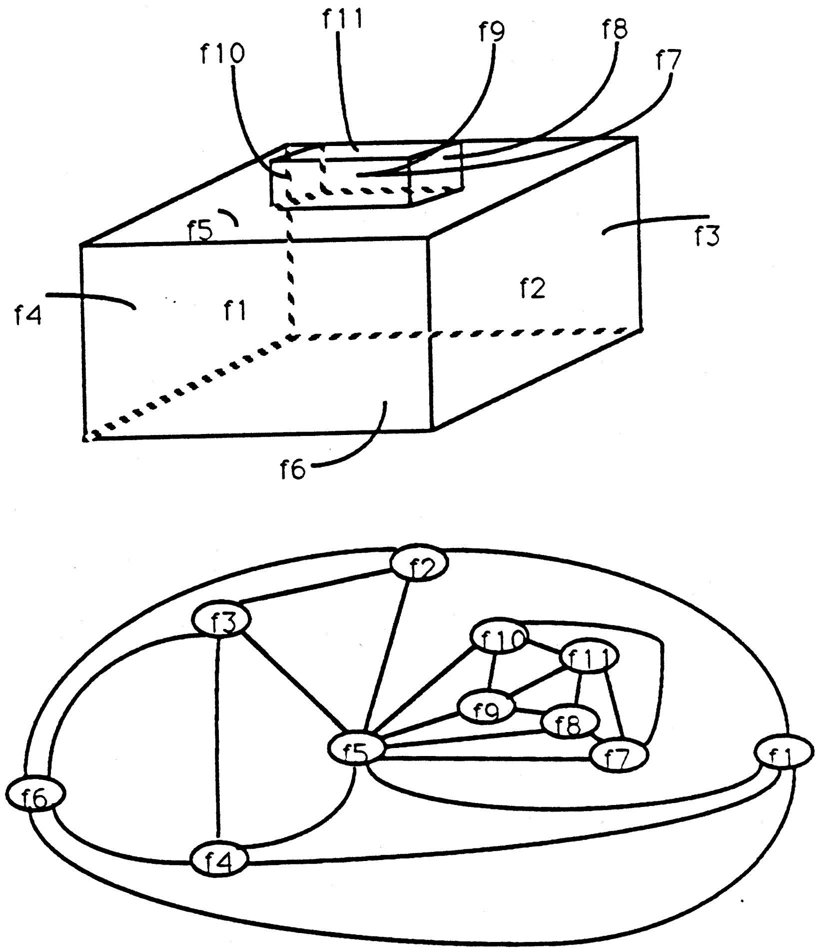

Figure 1. A Polyhedron with an articualtion point in its SDR

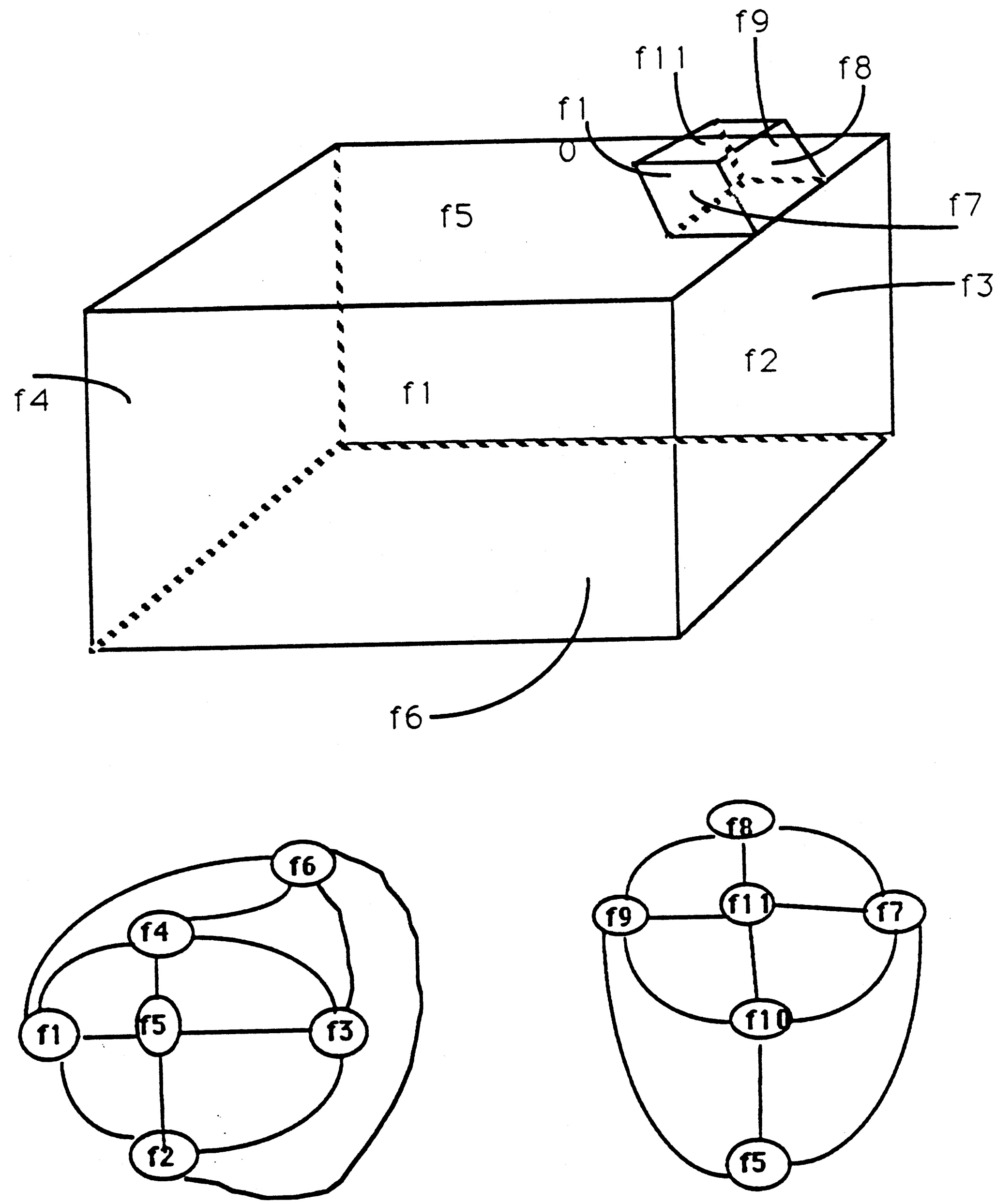

Figure 2. A Polyhedron and its incorrect 3-connected components

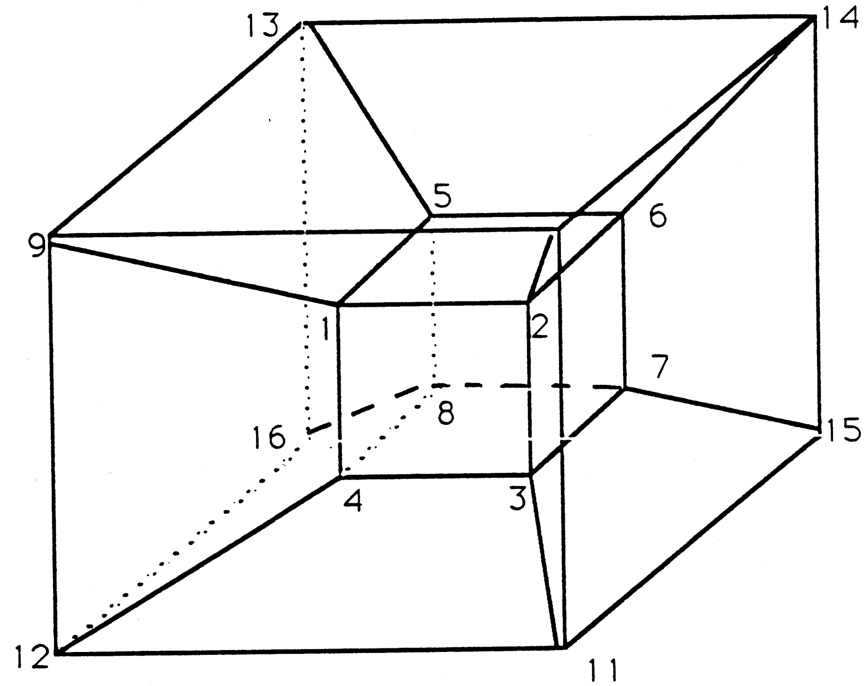

Figure 3. The hypercube connectivity graph

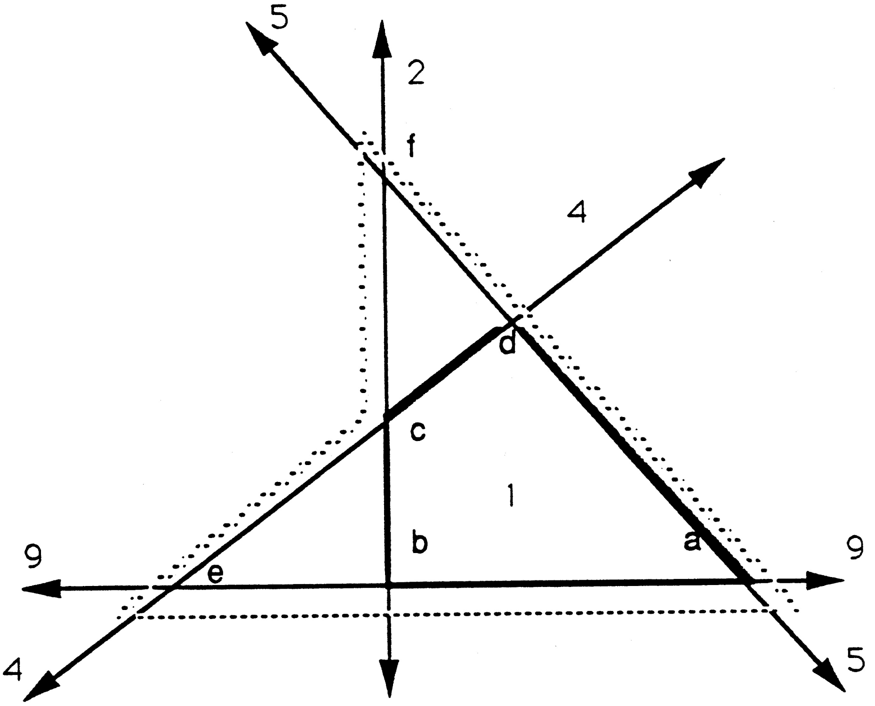

Figure 4. The two interpretations of face 1